Motives and Automorphic Forms: Applications of the Principle of Functoriality*

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Contents

0 Introduction **3**

1 Categories of Automorphic Representations **6**
1.1 Principle of Functoriality; Tannakian Operations 6
1.2 Categories of Automorphic and Admissible Representations 11
1.3 Formalism and Tannakian Problems 16
1.4 Abelian Categories of Isobaric Representations 19

2 Some Well-Known Cases of the Principle of Functoriality **21**
2.1 Automorphic Induction and Restriction 22
2.2 Symmetric Square 22
2.3 Functoriality Associated to the Exterior Square 22
2.4 Unitary Groups 23

3 Questions of Rationality **23**
3.1 The Field of Rationality of an Automorphic Representation 23
3.2 Unramified Representations: Rational Parametrization 27
3.3 The Action of $\mathrm{Aut}(\mathbb{C})$ on the Archimedean Type of Algebraic Representations 28
3.4 Conjectures and Verifications 29
3.5 The Case of Regular Representations 32

4 Motives and Algebraic Representations **48**
4.1 Motives 48

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4.2 Rational Algebraic Representations 48
4.3 Motives and Algebraic Representations 48
4.4 Consequences of Motivic Formalism 48
4.5 Hasse–Weil Conjecture and Reciprocity Theory for Algebraic Representations 48

5 The Case of Self-Dual Representations 48
5.1 Statement of the Theorem 48
5.2 Remarks 48
5.3 Sketch of Proof 48

0 Introduction

> Ne t'attarde pas à l'ornière des résultats.
> Seule les traces font rêver. (Do not linger in the rut of results.
> Only the traces make one dream.)
>
> René Char

The general conjectures of Langlands lead one to suspect a close relationship between the theory of automorphic forms and the arithmetic theory of algebraic varieties—more precisely, between automorphic representations and motives in the sense of Grothendieck. In his article for Corvallis, Langlands had sketched the idea of a "Tannakian" theory of automorphic forms: if one assumes that this theory exists, one is led to conjecture the existence of an "automorphic Galois group", which should be linked to the "motivic Galois group" postulated by Grothendieck.

The modern theory of automorphic forms, however, has led—since the introduction by Maass of the forms that bear his name—to the study of numerous objects which, unlike classical modular forms, have only a tenuous connection to arithmetic objects. This is already true of Hecke's "Grössencharaktere": in this case, Weil was led to define "Grössencharaktere of type A_0" — or algebraic Hecke characters—which are those that appear, for example, in the theory of complex multiplication.

The purpose of this article is to define, in the non-abelian case, the category of automorphic representations which should likewise be linked to "motives"—the non-abelian analogue of Hecke characters.

Once the "Langlands classification" for real groups is acquired, the definition of such a category imposes itself quite naturally (Borel had proposed it in [10b]). One realizes rather quickly, however, that the "obvious" definition does not fulfill two of the minimal required conditions: being amenable to the Tannakian formalism, and verifying the rationality properties that are satisfied by motives, and by algebraic Hecke characters.

The origin of the problem is that the Langlands classification, based on unitary induction, is not "algebraic": it does not respect the rationality of representations.

Taking the Tannakian formalism and rationality properties as our guides, we propose here a definition of the category of automorphic representations of $GL(n, \mathbb{A}_F)$, where $\mathbb{A}$ denotes the adeles of a number field F. We then show, using the extension to $GL(n)$ of the Eichler-Shimura theory, that these representations at least when their type at infinity is suitably regular—have the expected

rationality properties: if $\pi = \bigotimes_v \pi_v$ is an algebraic automorphic representation, assumed to be cuspidal, of $GL(n, \mathbb{A}_F)$, its finite part $\pi_f = \bigotimes_{v-\text{finite}} \pi_v$ is defined over a number field $E \subset \mathbb{C}$, and all conjugates of π_f by $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are again finite parts of cuspidal representations. Furthermore, if one admits Langlands's "functoriality principle", the category of algebraic representations should be a Tannakian category.

We then specify, in a fairly detailed manner, the conjectural relations between algebraic representations and motives. By construction, algebraic representations are endowed with structures that should be related to the various structures possessed by motives: Hodge theory; representations of Galois groups in l-adic cohomology. The purity properties of motives translate into the "Ramanujan Conjecture".

In a few cases, which we review, one already has a "motivic" interpretation of automorphic representations: these are essentially automorphic forms of arithmetic type on $GL(2)$, or Hecke characters. In the last paragraph, we show how to associate, under certain local hypotheses, compatible systems of l-adic representations to self-dual automorphic representations of $GL(n)$ over a totally real field. In particular, we prove the Ramanujan conjecture (at almost all places) for such representations.

The classically inclined reader may be surprised to see us study automorphic forms on $GL(n)$ rather than on reductive groups related to Hermitian symmetric domains: the holomorphic modular forms on the latter often admit an obvious "motivic" interpretation. The importance of the linear groups lies in the fact that, according to the functoriality principle, they "receive" all automorphic representations of classical groups. Thus, all automorphic forms on these groups that have a motivic interpretation must be linked to algebraic representations. The converse is not true: in fact, only motives satisfying very strong self-duality conditions appear, it seems, in the intersection cohomology of Shimura varieties (cf. §5.2.4). Moreover, considering $GL(n)$ allows one to use — at least conjecturally — the Tannakian formalism.

Here now is a more detailed description of the different paragraphs. Paragraph 1 describes the Tannakian formalism for $GL(n)$.

We have taken up the constructions of Langlands [38 f] by making them explicit: note that the fundamental result which is the basis of the formalism (Thm. 1.1), and which was assumed by Langlands, was proved by Jacquet and Shalika. It gives meaning, in particular, to the notion of an isobaric representation. The abelian category of isobaric representations is described in 1.4.3. Algebraic representations are defined in 1.2.3. We have described in detail how they translate in the "classical" cases (Hilbert modular forms).

Paragraph 2, which is utilitarian, describes certain cases of the functoriality principle—most of them established during the last ten years—which will be

useful to us.

Paragraph 3 studies the field of rationality of automorphic representations of $GL(n)$. We first prove that these are defined over their field of rationality (Prop. 3.1). We then explain how to transform the Langlands parametrization of unramified representations to make it rational. We then formulate the fundamental conjectures (Conjectures 3.7, 3.7', 3.8) on the rationality of algebraic representations (in particular, Conjecture 3.8 is the analogue of the Weil conjecture, proved by Waldschmidt, characterizing algebraic Hecke characters by their values). We verify that these conjectures are compatible with our definition of algebraic representations (§3.4.3) as well as with base change (§3.4.4).

We then prove Conjecture 3.7 for regular representations (Thm. 3.13). The proof relies on the Eichler-Shimura theory, reformulated in the adelic framework by Harder. The essential point consists in showing that the parabolic cohomology, considered as a subspace of the compact support cohomology of arithmetic groups, is stable under automorphisms of C. This is deduced from the self-duality properties of parabolic cohomology: the crucial result (Prop. 3.18) was also proved, in a different way, by Harder [30 a]. Here it is deduced from a simple but fundamental lemma concerning the image of compact support cohomology in L^2-cohomology (Lemma 3.17). An important consequence of the proof is that the parabolic cohomology (in the case of constant coefficients) is defined over Q: this is Theorem 3.19, which extends to $GL(n)$ a classical result of Eichler and Shimura.

Paragraph 4 is devoted to the "dictionary" relating motives and algebraic representations. We define rational algebraic representations there (§4.2), the analogue of "absolute" motives (without coefficients).

After introducing the (conjectural) Galois group associated to algebraic representations, and discussing its relations with Langlands's "automorphic Galois group", we give the fundamental conjecture concerning the relationship between motives and algebraic representations (4.3.2, Conjecture 4.5). It is specified in 4.3.3.

We provide some verifications concerning rationality and functorial behavior in 4.3.4, 4.3.4. In particular, we prove the fundamental Lemma 4.9, concerning the purity of algebraic representations at infinity; it is used in paragraph 3 and to define a priori the notion of weight for algebraic representations. In paragraph 4.5, we formulate a problem concerning the "reciprocal theory" for algebraic representations.

Finally, paragraph 5 describes how one can associate, if not motives, at least compatible systems of l-adic representations, to automorphic representations of $GL(n, \mathbb{A}_F)$ (F totally real) which are regular and self-dual. The proofs rely on recent results by Kottwitz.

During the writing of this article, I benefited from discussions with J.-L.

Waldspurger on the rationality questions of paragraph 3. I also had access to a manuscript by Harder [30 a] concerning the cohomology of arithmetic groups. During the conference, and afterwards, I benefited from explanations by D. Blasius and D. Ramakrishnan on the properties of the category of motives. If paragraph 4 contains no gross error, it is due to them. J. Tits described to me the Galois cohomology of unitary groups. Finally, R. Kottwitz explained to me, on several occasions, the miraculous properties of the groups described in paragraph 5 and the stabilization of the twisted trace formula. I am particularly grateful to him for having communicated to me before publication the essential results of [36]. Let them all be thanked here.

1 Categories of Automorphic Representations

In this chapter, we describe certain categories of automorphic representations of linear groups that should be amenable to a Tannakian formalism.

1.1 Principle of Functoriality; Tannakian Operations

In all that follows, we are interested in automorphic representations of $GL(n, \mathbb{A}_F)$ where F is a number field and $\mathbb{A} = \mathbb{A}_F$. Recall [12, §4.6] that an irreducible representation of $G(\mathbb{A}) = GL(L, \mathbb{A})$ is automorphic if it is isomorphic to an irreducible subquotient of the representation of $G(\mathbb{A})$ on the space of automorphic forms (for the problems this poses from the Tannakian viewpoint, cf.§1.3). We will denote by $\mathrm{Aut}(n)$ the class of automorphic representations of $GL(n, \mathbb{A}_F)$, $\mathrm{Aut} = \coprod_n \mathrm{Aut}(n)$.

Recall that cuspidal representations are those that appear in the space of cusp forms $\mathcal{A}_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}))$[12]. The theory of Eisenstein series shows (Langlands [38 e]) that every automorphic representation is isomorphic to a subquotient of an induced representation

$$\rho = \mathrm{Ind}_{P(\mathbb{A})=M(\mathbb{A})N(\mathbb{A})}^{G(\mathbb{A})}(\sigma \otimes 1), \tag{1.1}$$

where Ind denotes unitary induction from a parabolic subgroup $P = MN$ of G, and σ is a cuspidal representation of $M(\mathbb{A})$.

In all that follows, we will consider only, unless explicitly mentioned, the standard parabolics of $GL(n)$ formed by upper block-triangular matrices. If P is of type $(n_1, \ldots, n_r)$ where $n_1 + \cdots + n_r = n$, σ can be written in the form $\sigma_1 \otimes \cdots \otimes \sigma_r$, where σ_i is a cuspidal representation of $GL(n_i, \mathbb{A})$. The following result is due to Jacquet and Shalika.

Theorem 1.1 (33 a,b)**.** *Let $n = n_1 + \cdots + n_r = m_1 + \cdots + m_s$ be two partitions of n, S a finite set of places of F, containing the infinite places. For every*

$i = 1, \ldots, r$, let σ_i be a representation of $G_{n_i}(\mathbb{A})$ on an irreducible subspace V_i of $\mathcal{A}_{\text{cusp}}(G_{n_i}(F)\backslash G_{n_i}(\mathbb{A}))$; Similarly, let (τ_j, W_j) be a representation of $G_{m_i}(\mathbb{A})$ $(j = 1, \ldots, s)$. Assume σ_i and τ_j are unramified outside S. Suppose that for $v \notin S$, the diagonal matrices

$$\begin{pmatrix} t_{\sigma_1,v} & & \\ & \ddots & \\ & & t_{\sigma_r,v} \end{pmatrix} \quad , \quad \begin{pmatrix} t_{\tau_1,v} & & \\ & \ddots & \\ & & t_{\tau_s,v} \end{pmatrix}$$

where the $t_{\sigma_i,v}$ (resp. $t_{\tau_j,v}$) denote the Hecke matrices, are equal modulo $\mathfrak{S}_n$. Then $r = s$, and there exists a permutation $\alpha \in \mathfrak{S}_r$ such that $m_{\alpha_i} = n_i$ and the two submodules V_i and $W_{\alpha(i)}$ of $\mathcal{A}_{\text{cusp}}(G_{n_i}(F)\backslash G_{n_i}(\mathbb{A}))$ coincide.

The local analogue of this result is the Langlands classification. Suppose for a moment F is local. Let $P : n = n_1 + \cdots + n_r$ be a partition of n, and σ_i square-integrable representations modulo the center of $GL(n_i, F)$.

Let χ_i be the central character of σ_i on F^X. We have $\chi_i = |s_i|, s_i \in R$. Up to permutation, we can assume $s_1 \geq s_2 \geq \cdots \geq s_r$. The representation

$$\rho = \text{Ind}_{P(F)}^{G(F)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1) = \rho(\sigma_1, \ldots, \sigma_r) \tag{1.2}$$

then has a unique irreducible quotient, which is called its Langlands quotient; it appears with multiplicity 1 in ρ and thus in each of the representations $\rho(\sigma_{s.1}, \ldots, \sigma_{s.r})$ where $s \in \mathfrak{S}_r$. It is also called the Langlands subquotient of each of these representations (cf.[14, 50]). Note that the usual definition of the Langlands quotient makes use of induction from tempered representations. To translate this into our definition, one must know that for $GL(n)$ the parabolic inductions from unitary square-integrable representations are irreducible: [5, 4 ; 51 a].

According to Langlands, it is known that every irreducible representation of $GL(n, F)$ is isomorphic to the Langlands subquotient of a $\rho(\sigma_1, \ldots, \sigma_r)$ and that the σ_i are then well-determined up to permutation.

To extend the definition of the Langlands quotient to the local components of (A.1), one must introduce generic representations. Let

$$N_0 = \left\{ n = \begin{pmatrix} 1 & x_1 & * \\ \vdots & \ddots & x_{n-1} \\ 0 & \cdots & 1 \end{pmatrix} \right\}$$

ψ a non-trivial additive character of F, $\theta(n) = \psi(x_1 + \cdots + x_{n-1})$ the associated non-degenerate character of $N_0(F)$. Recall that a representation (π, V) of $GL(n, F)$ (F local) is generic if there is a distribution vector in its dual that

transforms under $N_0(F)$ according to θ. It is known that the local components of a cuspidal representation are generic [40, 49]. According to results of Kostant and Vogan in the real case, and Bernstein and Zelevinsky in the p-adic case, it is known that every irreducible generic representation can be written as the full induced representation $\rho(\sigma_1, \ldots, \sigma_r)$ (1.2) for square-integrable representations $\sigma_1, \ldots, \sigma_r$.

Let us then consider the local components of the representation ρ from (1.1), where $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$ is a tensor product of cuspidal representations of $GL(n, \mathbb{A}_F)$. The component at the place v is written as

$$\rho_v = \mathrm{Ind}_{P(F_v)}^{G(F_v)}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{r,v} \otimes 1). \tag{1.3}$$

Since the $\sigma_{i,v}$ are generic, they can in turn be written as induced representations, in the sense of (1.3), from square-integrable representations:

$$\rho_{i,v} = \mathrm{Ind}_{P_i(F_v)}^{G_{n_i}(F_v)}(\tau_{i,v}^1 \otimes \cdots \otimes \tau_{i,v}^r \otimes 1) \tag{1.4}$$

(full induction). We deduce, by induction by stages, that ρ_v is the full induction from the tensor product of the τ_i and therefore contains, with multiplicity 1, the corresponding Langlands subquotient.

Definition 1.2 (Langlands)**.** Let $\pi = \otimes \pi_v$ be an irreducible automorphic representation of $G(\mathbb{A})$. It is known that π is a subquotient of a representation (1.1) $\rho = \otimes \rho_v$. We say that π is *isobaric* if π_v is, at every place, the Langlands subquotient of ρ_v.

Note first that if ρ_v is unramified, its Langlands quotient is the associated spherical representation [31]. In particular, the isobaric representation associated with ρ does have the continuity property required of automorphic representations.

If $\sigma_1, \ldots, \sigma_r$ are cuspidal representations of $G_{n_i}(A_F)$, we will denote

$$\sigma_1 \boxplus \cdots \boxplus \sigma_r$$

the unique isobaric subquotient of the representation (1.1).

By construction, it does not depend, up to isomorphism, on the order of the σ_i. By definition, every isobaric representation is thus obtained; finally, by Theorem (1.1), two representations $\sigma_1 \boxplus \cdots \boxplus \sigma_r$ and $\sigma_1' \boxplus \cdots \boxplus \sigma_t'$ are isomorphic if and only if the σ_i are obtained by permutation of the σ_i'.

Examples: In (1.1), take P equal to the Borel subgroup, and

$$\rho = \mathrm{Ind}\left(|\ |^{\frac{-n+1}{2}}, |\ |^{\frac{-n+3}{2}}, \ldots, |\ |^{\frac{n-1}{2}} \right)$$

where $|\ |$ denotes the idele norm. The associated isobaric representation is the trivial representation.

More generally, let P be homogeneous of type $(m, \dots, m)$ $(n = am)$ and ω a cuspidal representation of $GL(m, \mathbb{A})$. The representation π such that, at every place v, π_v is the unique irreducible subquotient of

$$\mathrm{Ind}_{P_v}^{G_v}(\omega_v, \omega_v||, \dots, \omega_v||^{a-1}) \tag{1.5}$$

is isobaric and appears in the discrete spectrum [51 b]. Moeglin and Waldspurger [39] have recently shown that these representations constitute the entire discrete spectrum. We denote the unique irreducible quotient of (1.5) by $J(\omega_v, a)$ and the global product by $J(\omega, a)$.

Let us now recall how the functoriality principle is illustrated in the case of $GL(n)$:

• **Sums**: Let $n = n_1 + n_2 + \cdots + n_r$, and let π_i be automorphic representations of $GL(n_i, \mathbb{A})$. One can consider the induced representation

$$\rho = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi_1 \otimes \cdots \otimes \pi_r \otimes 1.$$

Its Hecke matrices, at unramified places, are

$$t_{\rho,v} = \begin{pmatrix} t_{\pi,v} & & \\ & \ddots & \\ & & t_{\pi_r,v} \end{pmatrix}.$$

In other words, the Hecke matrices are determined by the homomorphism of L-groups

$${}^L G_{n_1}^0 \times \cdots \times^L G_{n_r}^0 \to^L G_n^0$$

sending $GL(n_1) \times \cdots \times GL(n_r)$ to $GL(n)$ by blocks.

If the representations π_i are **isobaric**, one can consider the unique isobaric subquotient of ρ (it exists by the same arguments used for cuspidal π_i). We denote it by

$$\pi = \pi_1 \boxplus \cdots \boxplus \pi_r.$$

• **Products**. Let $n = ab$, π_1 an automorphic representation of $GL(a, \mathbb{A})$, π_2 an automorphic representation of $GL(b, \mathbb{A})$.

Conjecture 1.3. *There exists an automorphic representation π of $GL(ab, \mathbb{A})$ such that, for v outside a finite set S of places:*

$$t_{\pi,v} = t_{\pi_1,v} \otimes t_{\pi_2,v}. \tag{1.6}$$

(the tensor product of two diagonal matrices defines a diagonal matrix, which depends only on π_1 and π_2 modulo $\mathfrak{S}_n$).

This conjecture deserves to be called the main conjecture for $GL(n)$. Note that it would suffice to prove it for cuspidal representations. One can refine it for isobaric representations: let π_1, π_2 be isobaric. If the conjecture is true for π_1 and π_2, there exists a unique **isobaric** representation of $GL(n, \mathbb{A}_F)$ satisfying (1.6).

Definition 1.4. If the isobaric representation of $GL(n, \mathbb{A}_F)$ satisfying (1.6) exists, it is called the exterior tensor product of π_1 and π_2; it is denoted $\pi_1 \boxtimes \pi_2$.

On the L-group side, this operation corresponds of course to the homomorphism

$$ {}^L G_a^0 \times {}^L G_b^0 \to {}^L G_n^0$$

given by the tensor product: $GL(a) \times GL(b) \to GL(n)$.

• **Powers**. More generally, let r be a rational representation of $GL(n)$, of degree N. If $G = GL(n)$ and $H = GL(N)$, we consider r as a homomorphism ${}^L G^0 \to {}^L H^0$. We extend it to the L-groups, trivially on the Galois groups. According to the functoriality principle, we similarly deduce a conjectural map, sending isobaric representations τ_n of $GL(n, \mathbb{A})$ to those, π_N, of $GL(N, \mathbb{A})$, and determined by the fact that

$$ t_{\pi_{N,v}} = r(t_{\pi_{n,v}})$$

as a conjugacy class in $GL(N, \mathbb{C})$.

Important special cases are the symmetric powers S^i, exterior powers Λ^i of the standard representation of $GL(n)$.

More generally, one can consider rational representations of $G = GL(n_1) \times \cdots \times GL(n_r)$. This recovers the case of products.

• **Base change**. Let E/F be a number field extension. We consider $GL(n)/E$ as an F-group by restriction of scalars. Its L-group is then [10a,§5]

$$ {}^L H = GL(n, \mathbb{C}) \times \cdots \times GL(n, \mathbb{C}) \times \mathfrak{g}_F;$$

there is one copy of $GL(n, \mathbb{C})$ for each embedding $E \to \bar{F}$ over F, and $\mathfrak{g}_F$ acts by permutation. This yields two functorial operations:

(a) Restriction. Setting $G = GL(n)/F$, we have a diagonal embedding ${}^L G \to {}^L H$ which sends ${}^L G^0$ diagonally into ${}^L H^0$. Dually, we deduce a restriction operation $\rho_{F/E}$ sending isobaric representations of $G(\mathbb{A}_F)$ to those of $G(\mathbb{A}_E)$.

(b) Induction. Let $d = [E : F]$, $G_{dn} = GL(dn)/F$. We have an embedding

$$ {}^L H \to {}^L G_{dn}$$

sending the matrices of ${}^L H^0$, diagonally by blocks, to matrices of $GL(dn, \mathbb{C})$. We extend it to ${}^L H^0$ by having the elements of $\mathfrak{g}_F$ act by permutation matrices.

Dually, we deduce a (conjectural) operation of **automorphic induction** ι_E^F sending isobaric representations of $GL(n, \mathbb{A}_E)$ to those of $GL(dn, \mathbb{A}_F)$.

For the description, in the case where E/F is cyclic of prime order, of $\rho_{F/E}$ and ι_E^F in terms of Hecke matrices, see [18 d].

Base change commutes with the "geometric" operations $\boxplus$ and $\boxtimes$ (representations being considered up to isomorphism).

In Chapter 2, we will review some cases where the functoriality principle has been established. It will be convenient to first describe certain categories of representations.

1.2 Categories of Automorphic and Admissible Representations

We now aim to define certain categories of automorphic representations, which can be accessible to the Tannakian formalism (cf. §1.3). Here we content ourselves with defining the objects. The sets of automorphic representations considered must be stable under the Tannakian operations $\boxplus$ and $\boxtimes$ as well as under the adjunction which associates to π its contragredient $\tilde{\pi}$. For these to be well-defined, we first restrict ourselves to isobaric representations.

1.2.1 The category $Isob$.

Its objects are the isobaric representations. The direct sum $\pi \boxplus \pi'$ is well-defined there; the tensor product is (up to isomorphism) modulo the Main Conjecture.

We denote by $Isob(n)$ the set of objects of degree n (representation of $GL(n, \mathbb{A})$); similarly for other categories.

1.2.2 The categories $Temp$ and IC.

The categories Temp are the irreducible automorphic representations of $GL(n, \mathbb{A})$- for the various n- whose components at all places are **tempered**.

Lemma 1.5. *If $\pi \in Temp(n)$, then π is isobaric.*

As an automorphic representation, π is indeed a subquotient of $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$ for cuspidal representations σ_i of $GL(n_i, \mathbb{A})$. At almost all places, π_v is unramified. The representations $(\sigma_i)_v$ are therefore also unramified, and π_v is, by parabolic induction, the Langlands subquotient of $\mathrm{Ind}_{B(F_v)}^{G(F_v)}(\chi_1, \ldots, \chi_n)$ where B is the Borel subgroup, and the χ_i are unramified. Since π_v is tempered, the uniqueness in the Langlands classification implies that the χ_i are unitary and the induced representation is irreducible. We deduce that the σ_i must be **unitary** (at almost all places, and hence everywhere); therefore $\pi \cong \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$ is isobaric.

Recall the:

Conjecture 1.6 ("Ramanujan Conjecture"). *If π is a unitary cuspidal representation of $GL(n, \mathbb{A}_F)$, then the local components of π are tempered.*

It is clear that the direct sum of two representations in Temp is again tempered. Lemma 1.5 actually shows that if π is an automorphic representation whose local components are almost all tempered, it is a sum (for $\boxplus$ of unitary cuspidal representations. If we admit Conjecture 1.6, and of course Conjecture 1.3, we see that Temp is preserved by the tensor product.

Let IC be the category of representations unitarily induced from unitary cuspidal representations. The Ramanujan conjecture would imply that IC is included in $Temp$. Conversely, the proof of Lemma 1.5 shows that $Temp$ is included in IC. It is often useful (cf. [2, ch. III]) to consider the category IC directly, which is simply described without waiting for the solution of the Ramanujan conjecture. It consists of isobaric representations and is stable under $\boxplus$.

The Ramanujan conjecture has an interesting consequence for representations in $Isob$:

Lemma 1.7 (Global Langlands quotient). *Assume Conjecture 1.6. Then, if $\pi \in Isob(n)$, π is the unique irreducible quotient of a representation*

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$$

where the σ_i are cuspidal representations of $G_{n_i}(\mathbb{A})$, P is the parabolic of type $(n_1, \ldots, n_r)$, $\sigma_i \cong \sigma_i^0 |\ |^{s_i}$ where σ_i^0 is unitary and $s_i \in \mathbb{R}$, $s_1 \geq s_2 \geq \cdots \geq s_r$.

This follows immediately from formulas (1.3) and (1.4) since the σ_i^0 are then tempered.

1.2.3 The category *Alg*.

The theory of automorphic forms on $GL(1)$ - i.e., characters of the idele class group - shows that the categories introduced so far are too vast to be linked to arithmetic objects. We now wish to define the category of representations of $GL(n)$ that plays a role analogous to that of Grössencharaktere of type A_0, or algebraic Hecke characters, of Weil. As we will see (cf. also §3), this leads to modifying the definition of $\boxplus$.

Recall that according to Langlands [38 d, 51] there is a canonical bijection between irreducible admissible representations of $GL(n, F)$ ($F = \mathbb{R}$ or $\mathbb{C}$), modulo equivalence, and complex representations of degree n, with semisimple image, up to equivalence, of the Weil group W_F. We have an exact sequence

$$1 \to W_{\mathbb{C}} \to W_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{C}/\mathbb{R}} \to 1$$

and $W_{\mathbb{C}}$ is canonically isomorphic to $\mathbb{C}^\times$ [53]. It is sometimes convenient to consider $\mathbb{C}^\times$ as the set of real points of $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(G_m)$.

We denote by r the representations of the real or complex Weil group. Let us emphasize that the parametrization we use is that of Langlands. In particular, it sends **tempered** representations of $GL(n, F)$ to representations with bounded image of W_F.

Let $\pi = \otimes_v \pi_v$ be an irreducible representation of $GL(n, \mathbb{A})$. Let v be an infinite place of F, and fix an isomorphism of F_v with $\mathbb{R}$ or $\mathbb{C}$. The Langlands classification associates to v a representation r_v of W_F. By restriction in the real case, we deduce a representation - still denoted r_v - of $\mathbb{C}^\times$; in the complex case, r_v is the representation of $\mathbb{C}^\times$ defined by the choice of isomorphism.

Finally, let $|\ |_{\mathbb{C}}$ be the complex absolute value on $\mathbb{C}^\times$.

Definition 1.8. $Alg(n)$ is the set of isobaric representations of $GL(n, \mathbb{A})$, $\pi = \otimes \pi_v$, such that, for every infinite place v, r_v is equal to $\chi_1 \otimes \cdots \otimes \chi_n$ where χ_i is a character of $S(\mathbb{R}) = \mathbb{C}^\times$ such that $\chi_i |\ |_{\mathbb{C}}^{\frac{1-n}{2}}$ is an algebraic character.

In other words, $\chi_i(z) = z^{p_i + \frac{n-1}{2}} (\bar{z})^{q_i + \frac{n-1}{2}}$, with p_i, $q_i \in \mathbb{Z}$.

Remark. Let v be an infinite place of F. Let H be the maximal (diagonal) torus of G. If v is complex, let $\mathfrak{h}_0 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathrm{Lie}\, H)$; if v is real, $\mathfrak{h}_0 = \mathrm{Lie}\, H$. Let $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$. We have $\mathfrak{h} \cong \mathbb{C}^n \times \mathbb{C}^n$ (complex case), $\mathfrak{h} \cong \mathbb{C}^n$ (real case), the complex Weyl group W operating as $\mathfrak{S}_n \times \mathfrak{S}_n$ (resp. $\mathfrak{S}_n$). According to Harish-Chandra, the center 3 of $U(\mathfrak{g})$, where $\mathfrak{g}_0 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{Lie}\, G$ or $\mathrm{Lie}\, G$ depending on the case, and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$, is identified with $S(\mathfrak{h})^W$.

The infinitesimal character of π_v is then identified with the element $(p, q) \in \mathbb{C}^{2n}$ (complex case), or with $(p_1, \ldots p_{2r}, p'_{2r+1}, \ldots, p'_n) \in \mathbb{C}^n$ if r_v is of the form $(z^{p_1} \bar{z}^{p_2}, z^{p_2} \bar{z}^{p_1}, \ldots, (z\bar{z})^{p'_{2r+1}}, \ldots, (z\bar{z})^{p'_n})$ with $p_i \neq p_j$ (real case). Laumon pointed out to me that - modulo the center - the representations of Alg are thus associated to automorphic forms satisfying differential equations associated to a character of $\mathfrak{Z}$ which is integral for the **integral** structure on $\mathfrak{h}$ deduced from a Chevalley basis.

Example. $GL(2)$ over a real field.

It will be useful to describe here how the definition of Alg translates in the classical cases.

Let's start with the case of $GL(2)$ over $\mathbb{Q}$. Let f be a parabolic form of weight $k \in \mathbb{Z}$ on the Poincaré half-plane for the group $\Gamma_0(n)$, associated to a character ψ of $(\mathbb{Z}/N\mathbb{Z})^\times$:

$$f\left(\frac{az+b}{cz+d}\right) = \psi(a)^{-1} j(\gamma, z)^k f(z), \quad \gamma \in \Gamma_0(N),$$

where $j(\gamma, z) = cz + d$. The weight k lift [26] associates to it a parabolic form φ_f on $GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})$, whose central character is the Artin character ω of $\mathbb{A}^\times$ associated to ψ. We assume that f is an eigenfunction of the Hecke operators for $p|N$, and is holomorphic if $k > 0$, antiholomorphic for $k < 0$, and an eigenfunction of the hyperbolic Laplacian:

$$\Delta f = \lambda f, \quad \Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

for $k = 0$; finally we can assume that f is a newform. More generally, one can twist the weight k lift by a character $|\det g|^s$ of the determinant, thus replacing ω by $\omega|\ |^{2s}$.

Holomorphic case $k \geq 2$. If f is holomorphic, φ_f then generates a representation $\pi = \pi_\infty \otimes \pi_f$ of $GL(2,\mathbb{A}_\mathbb{Q})$ such that π_∞ is associated, by the Langlands correspondence, to a pair of characters $(z^p\bar{z}^q, z^q\bar{z}^p)$ $(p \neq q)$ of $\mathbb{C}^\times$. It is algebraic if $p = p_0 + \frac{1}{2}$, $q = q_0 + \frac{1}{2}$ with p_0, $q_0 \in \mathbb{Z}$; its central character on $\mathbb{R}_+^\times$ is $x \mapsto x^m$, with $m = p + q = p_0 + q_0 + 1 \equiv p_0 - q_0 + 1 \pmod 2$; the weight of f is $k = |p - q| + 1 = |p_0 - q_0| + 1$. Therefore:

The lifts π of f that are algebraic are those with central character $x \mapsto x^m(\mathbf{sgn}\ x)^\epsilon$ with $m \equiv k \pmod 2$.

The antiholomorphic case $(k \leq -2)$ is treated similarly.

Forms of weight 0 or 1. If f is of weight 0 or ± 1, the associated representation π is such that $\pi_\infty = \xi_1 \boxplus \xi_2$, where $\xi_i = |x|^{s_i}(\mathrm{sgn}\ x)^{\epsilon_i}$ are characters of $\mathbb{R}^\times$. It is algebraic if $s_i \in \frac{1}{2} + \mathbb{Z}$. Since it is unitary impelled by the center, we have $s_1 - s_2 \in i\mathbb{R} \bigcup\ ]-1, +1[$. We must therefore have $s_1 = s_2 \in \frac{1}{2} + \mathbb{Z}$. If $\epsilon_1 = \epsilon_2$, f is a form of weight 0, with Laplacian eigenvalue $\lambda = -(s - \frac{1}{2})(s + \frac{1}{2})$ where $s = s_1 - s_2$. Thus $\lambda = \frac{1}{4}$. Finally, the central character of π_∞ on $\mathbb{R}_+^\times, x \mapsto |x|^{s_1+s_2}$, is of the form x^m with $m \in 1 + 2\mathbb{Z}$:

If f is a Maass form, the associated representation π can only be algebraic if $\lambda = \frac{1}{4}$. If f is of weight 0, -1, or 1, the algebraic lifts are the representations π with central character $x \mapsto x^m(\mathbf{sgn}\ x)^\epsilon$ with $m \equiv 1 \pmod 2$.

Now suppose F is **totally real** of degree r and let f be a parabolic form of weights $k_1, \ldots, k_r$ $(k_i \in \mathbb{Z})$. We also have the weight $k = (k_1, \ldots, k_r)$ lift of f; it generates an irreducible cuspidal representation π of $GL(2,\mathbb{A}_F)$ under the previous assumptions on f. It can be twisted by characters of the form $|\det g|^s$ (recall also that the algebraic Hecke characters of $\mathbb{A}_F^\times$ are of the form $\omega(x)|x|^m$, $m \in \mathbb{Z}$, ω an Artin character).

That being said, if $\pi_\infty = \bigotimes_{v_i \text{ infinite}} \pi_i$ is algebraic, π_i must be associated to $z^{p_i}\bar{z}^{q_i}$, $z^{q_i}\bar{z}^{p_i})$ with $p_i = p_{i,o} + \frac{1}{2}$, $q_i = q_{i,o} + \frac{1}{2}$, $p_{i,o}, q_{i,o} \in \mathbb{Z}$. Moreover, since π, up to twist by a character, is unitary, we further see that $p_{i,o} + q_{i,o} = w$ is

independent of i; as $|k_i| = |p_{i,o} - q_{i,o}| + 1$ if $p_{i,o} \neq q_{i,o}, k_i = 0, 1, -1$ otherwise, we have the following result.

Set $\ell_i = k_i$ if $|k_1| \geq 2, \ell_i = 1$ if $|k_i| \leq 1$.

The representation π associated to f can only be algebraic if $\ell_i \equiv \ell_j \pmod 2$ for all i, j. If this condition is satisfied, the algebraic lifts π of f are those with central character $\omega(x)|x|^m$, ω being an Artin character and $m \equiv \ell_i \pmod 2$.

For the moment, Alg is not stable under the sum. We define a new operation $\overset{T}{\boxplus}$ using a Tate twist. If $\pi \in Isob(n)$, let $\pi|\ |^s$ be the representation $\pi(g)|\det g|^s$, where $|\ |$ denotes the idele norm.

Definition 1.9. Let $\pi_1 \in Isob(n_1)$, $\pi_2 \in Isob(n_2)$. We define

$$\pi_1 \overset{T}{\boxplus} \pi_2 = (\pi_1|\ |^{\frac{1-n_1}{2}} \boxplus \pi_2|\ |^{\frac{1-n_2}{2}})\ |\ |^{\frac{n_1+n_2-1}{2}}$$

By construction, the operation $\overset{T}{\boxplus}$ preserves the category $Alg = \coprod Alg(n)$ since, at infinite places, $\overset{T}{\boxplus}$ translates to the sum of representations of the Weil group.

Modulo Conjecture 1.3, we similarly have:

Definition 1.10. We set

$$\pi_1 \overset{T}{\boxtimes} \pi_2 = (\pi_1|\ |^{\frac{1-n_1}{2}} \boxtimes \pi_2|\ |^{\frac{1-n_2}{2}})\ |\ |^{\frac{n_1 n_2-1}{2}}.$$

It also preserves the category Alg. Moreover, the properties of $\boxplus$ and $\boxtimes$ implied by the Tannakian formalism (§1.3) are preserved: for example, $(\pi_1 \overset{T}{\boxplus} \pi_2) \overset{T}{\boxplus} \pi_3$ is isomorphic (in fact canonically isomorphic, by parabolic induction) to $\pi_1 \overset{T}{\boxplus} (\pi_2 \overset{T}{\boxplus} \pi_3)$, $\overset{T}{\boxplus}$ and $\overset{T}{\boxtimes}$ satisfy (conjecturally and up to isomorphism) the usual compatibilities...

In contrast to the Langlands sum, the operation $\overset{T}{\boxplus}$ does not commute with passage to the contragredient.

Definition 1.11. If $\pi \in Aut(n)$, its Tate dual $\check{\pi}$ is the representation $\tilde{\pi}|\ |^{n-1}$.

It is clear that $\pi \to \check{\pi}$ is an involution. One verifies that $(\pi_1 \overset{T}{\boxplus} \pi_2)\check{} = \check{\pi}_1 \overset{T}{\boxplus} \check{\pi}_2$ if $\pi_1, \pi_2 \in Isob(n_1), Isob(n_2)$.

The decomposition of isobaric representations takes a new form in Alg:

Lemma 1.12. *If $\pi \in Alg(n)$, one can write*

$$\pi \cong \pi_1 \overset{T}{\boxplus} \pi_2 \ldots \overset{T}{\boxplus} \pi_r,$$

where $\pi_i \in Alg^0(n_i)$ (the set of cuspidal representations of $Alg(n_i)$); the (π_i, n_i) are unique up to permutation.

This follows directly from the definitions.

1.2.4 The categories Tan and Art.

The category Tan consists of representations of Alg which, by automorphic restriction (§1.1) to a finite extension E/F, become sums of (algebraic) Hecke characters of $GL(1, \mathbb{A}_E)$:

$$\rho_{F/E}\pi \cong \chi_1 \overset{T}{\boxplus} \chi_2 \overset{T}{\boxplus} \dots \overset{T}{\boxplus} \chi_n,$$

$\chi_i \in Alg(1, E)$.

The category Art consists of representations of Alg such that at infinite places, with the notations 1.2.3, $r_v = |\ |_{\mathbb{C}}^{\frac{n-1}{2}} \oplus \dots \oplus |\ |_{\mathbb{C}}^{\frac{n-1}{2}}$.

These categories are stable under $\overset{T}{\boxplus}$ and, conjecturally, under $\overset{T}{\boxtimes}$.

1.2.5 Self-dual subcategories.

In each of the considered categories, one can consider the set of representations stable under the duality ($\pi \mapsto \tilde{\pi}$ for $Isob$ or $Temp$, $\pi \mapsto \tilde{\pi}$ for the subcategories of Alg). One then obtains subcategories (stable under $\boxplus$ or $\overset{T}{\boxplus}$ and, conjecturally, under $\boxtimes$) of representations. We will see (§5) that for these the correspondence between motives and automorphic representations seems more accessible.

1.3 Formalism and Tannakian Problems

1.3.1

We assume the reader is somewhat familiar with the formalism of Tannakian categories due to Grothendieck and Saavedra Rivano (cf. [45, 21]). Let us simply recall that a neutral Tannakian category is an abelian k-linear category (for a field k) $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \to \mathcal{C}, (X, Y) \mapsto X \otimes Y$, with certain data (associativity and commutativity constraints) expressing the usual properties of the tensor product, and a k-linear functor (exact, faithful, compatible with $\otimes$) $\mathcal{C} \to \mathrm{Vec}_k$, where Vec_k is the tensor category of vector spaces over k.

In [38 f], Langlands expresses the hope that one can equip the set $Isob$ of isobaric representations with a structure of a Tannakian category. At the level of objects, up to isomorphism, the direct sum should be given by the operation $(\pi, \pi') \to \pi \boxplus \pi'$ and the tensor product by $(\pi, \pi') \to \pi \boxtimes \pi'$ (recall that the

existence of the latter is conjectural!). The field k should here be the field of complex numbers.

As Langlands remarks "the attempt [to define Tannakian categories of automorphic representations] may be vain but the prize to be won is so great that one cannot refuse to hazard it". Indeed, even assuming the Main Conjecture (§1.1) is proven, the problems to be solved seem formidable.

The first problem to solve consists of equipping the category $Isob$ - or the other categories of §1.2 - with a structure of a $\mathbb{C}$-linear **abelian** category. In §1.4, we propose a definition of $\mathrm{Hom}_{Isob}(\pi, \pi')$ for π, π' in a suitable subset of Isob. We will see that, even to define $\mathrm{Hom}_{Isob}(\pi, \pi')$, one cannot work with representations isomorphic to (isobaric) subrepresentations of $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$. One is forced to restrict to a subset, containing up to $G(\mathbb{A})$-isomorphism all representations of $Isob$ (one then verifies that one indeed obtains an abelian category).

The problem of defining a suitable subset of Isob (or the other categories of 1.2) becomes even more crucial if one considers the tensor product. The definition of Tannakian categories requires that one can exhibit, for $\pi, \pi' \in Isob$, their tensor product $\pi \boxtimes \pi'$, a well-defined object of Isob. Even assuming Conjecture 1.3, it seems implausible that one can naturally define an object $\pi \boxtimes \pi'$ if π, π' are only **isomorphic** to representations appearing in $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$.

On the other hand, one can hope that a suitable multiplicity one theorem (cf. Th. 1.1) implies, for example, the existence of a unique representation (isomorphic to) $\pi \boxtimes \pi'$ and appearing as a submodule of $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$. We see that this leads to searching for **models** of automorphic representations, i.e., well-determined representatives of the isomorphism classes of (isobaric) automorphic representations.

1.3.2 Whittaker models

Let ψ be a non-trivial character of $F\backslash\mathbb{A}_F$, θ the character of

$$N_0(\mathbb{A}) = \left\{ \begin{pmatrix} 1 & x_1 & & * \\ & \ddots & & \\ & & & x_{n-1} \\ & & & 1 \end{pmatrix} \right\}$$

defined by $\theta(n) = \psi(x_1 + \cdots + x_{n-1})$. If $\pi = \otimes\pi_v$ is **generic**, π has a unique representative $W(\pi)$ in the space $W(\theta)$ of functions on $G(\mathbb{A})$ transforming on the left according to θ [40, 49]. This applies in particular to representations in $Temp$ (or IC).

Note the local analogue: assume F is a non-archimedean local field; ψ is then a non-trivial additive character of F, θ the associated character of $N_0(F)$. By definition, a generic representation π of $G(F)$ has a representative - then unique

- in the space W_θ of functions on $G(F)$ such that $f(ng) = \theta(n)f(g)$ $(n \in N(F))$ [28, 33 a] (cf. §1.1).

1.3.3 Automorphic models.

Let $\pi \in Aut^0(n)$. According to Theorem 1.1, there exists a unique submodule of $A_{\text{cusp}}(G(F)\backslash G(\mathbb{A}))$ isomorphic to π. One may wonder if there is an analogous result for representations in $Isob$ or $Temp$. The author does not know, but one can note the following facts:

1. The theory of Eisenstein series [38 a] shows that representations in IC (in $Temp$, modulo the Ramanujan conjecture) appear with multiplicity 1 in the Hilbert space decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$.

2. For the (isobaric) representations Speh introduced in §1.1, there is a theorem due to Moeglin and Waldspurger. Fix a central character $\chi : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$. Let $L^2_{\text{dis}}(G(F)\backslash G(\mathbb{A}), \chi)$ be the discrete part of the space of functions on $G(F)\backslash G(\mathbb{A})$, transforming according to χ under $Z(\mathbb{A}) \cong \mathbb{A}^\times$, and square integrable modulo the center.

Theorem 1.13 (cf [39]). *If $\pi \cong J(\omega, a)$ is a Speh module (notations 1.1) with central character χ, π appears in $L^2(G(F)\backslash G(\mathbb{A}), \chi)$ with multiplicity 1.*

1.3.4 Principal models.

Let π be an isobaric representation of $GL(n, \mathbb{A})$. One can write it in the form $\sigma_1 \boxplus \cdots \boxplus \sigma_r$, where the σ_i are cuspidal representations of $GL(n_1, \mathbb{A})$ (§1.1). Since every representation σ_i has a unique model in $\mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbb{A}))$, one obtains - for any choice of the order of the σ_i - a unique model of π, the unique subquotient of

$$\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$$

of type π. There are as many "models" (sic) as possible permutations of the σ_i. These models will be called **principal models**.

1.3.5

Let us conclude this paragraph by recalling that the irreducible admissible representations of $GL(n, F)$, where F is **local** should be parameterized by complex representations of degree n of the Weil-Deligne group WD_F [53, 38 f]. In particular, the operations $\boxplus$ and $\boxtimes$ should be defined (at least up to isomorphism) for local representations. In the archimedean case, where the classification is known, this suggests the following problem:

Problem 1.14 ($F = \mathbb{R}$ or $\mathbb{C}$.). *Define a suitable subcategory of the category of irreducible admissible representations of $GL(n, F)$ and equip it with a structure of a Tannakian category (for a description of a small subcategory of which one can make at least an* ***abelian*** *category, cf. 1.4.2).*

1.4 Abelian Categories of Isobaric Representations

1.4.0

In this paragraph, we construct a (small) abelian category of isobaric representations, where we describe the Hom spaces which should be those of the Tannakian category. It contains an object isomorphic to every isobaric representation; it is simply described using the principal models. There is an analogous local construction. This category has the defect that the application $\boxtimes$ (or at the object level) or the global-local restriction cannot be defined in it. In the last paragraph, we briefly indicate the construction of another category, equivalent, where these applications are defined. It is clear that these constructions are insufficient with regard to the hopes of §1.3: in particular, we have not shown here how to make the operation $\boxtimes$, or even the global-local restriction, functional.

1.4.1

We call $Isob_1 = Isob_1(F)$ the category whose objects are the principal models (1.3.4). An object $\pi \in Isob_1$ is identified with the data of a partition $n = n_1 + \cdots + n_r$ and r cuspidal representations σ_i of $G_{n_i}(\mathbb{A})$. We will call here **cuspidal representation** a subrepresentation of $\mathcal{A}_{\text{cusp}}(G_{n_i}(F)\backslash G_{n_i}(\mathbb{A})$: it is uniquely defined by its isomorphism class. We denote $\text{Hom}_{\mathcal{G}}(\pi, \pi')$ the space of homomorphisms from π to π' in $Isob_1$ (see §4 for the notation).

Definition 1.15. (1) If σ, τ are cuspidal, we set

$$\text{Hom}_{\mathcal{G}}(\sigma, \tau) = 0 \quad \text{if} \quad \sigma \neq \tau$$

$$\text{Hom}_{\mathcal{G}}(\sigma, \sigma) = \text{Hom}_{G(\mathbb{A})}(\sigma, \sigma) = \mathbb{C}$$

(2) We set $\text{Hom}_{\mathcal{G}}(\sigma_1 \boxplus \cdots \boxplus \sigma_r, \tau_1 \boxplus \cdots \boxplus \tau_s) = \bigoplus_{i,j} \text{Hom}_{\mathcal{G}}(\sigma_i, \tau_j)$.
In part (1), note that $\text{Hom}_{G(\mathbb{A})}(\sigma, \sigma)$ is canonically isomorphic to $\mathbb{C}$!

Composition. The morphisms compose in the obvious way, like matrices.

We define the zero object as the unique cuspidal representation of $GL(0, A_F)$.

Let $\pi = \sigma_1 \boxplus \cdots \boxplus \sigma_r, \pi' = \tau_1 \boxplus \cdots \boxplus \tau_s$ be two elements of $Isob_1$, with σ_i, τ_j cuspidal. Since $\pi \boxplus \pi'$ is, by definition, equal to $\sigma_1 \boxplus \cdots \boxplus \sigma_r \boxplus \tau_1 \boxplus \cdots \boxplus \tau_s$, one easily verifies that it is both a direct sum and a direct product of π and π'. The notation $\boxplus$ is therefore justified.

One easily verifies the existence of kernels and cokernels (it is convenient to note that every $\pi \in Isob_1$ can be written uniquely as a direct sum in $Isob_1$ of isotypic representations, i.e., of the form $\sigma \boxplus \cdots \boxplus \sigma$; one is reduced to the existence of kernels and cokernels in the category of vector spaces equipped with a basis). One similarly sees that the category thus defined is abelian.

Note that, even assuming Conjecture 1.3, the tensor product is not defined in $Isob_1$. In a category equipped with tensor products, one defines the dual of an object using the internal Hom: $\mathrm{Hom}(X, Y)$ ([21, p. 110]). In the case of $Isob_1$, one can define the dual of $\pi = \sigma_1 \boxplus \cdots \boxplus \sigma_r$ as the object $\tilde{\sigma}_1 \boxplus \cdots \boxplus \tilde{\sigma}_r$; let us denote it $\tilde{\pi}$. One easily verifies that one thus obtains a contravariant involution of $Isob_1$.

1.4.2

If F is a local field, the local analogue of $Isob_1$ is defined as follows. First, there is no need to impose restrictions on local representations: one considers all irreducible admissible representations. Such a representation of $G_n(F)$ is isomorphic to the unique Langlands subquotient of an induced representation

$$\mathrm{Ind}_{P(F)}^{G(F)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1)$$

(formula 1.2) where the σ_i are square-integrable. We denote this subquotient $\sigma_1 \boxplus \cdots \boxplus \sigma_r$.

To have the analogue of the principal models (1.3.4), one must fix models of the square-integrable representations. This is done using the local Whittaker model (cf. 1.3.2, [28]). One then has a unique model for each square-integrable representation; one then defines the abelian category $Isob_1(F)$, as in 1.4.1, using the principal models (note that one could also have used the Whittaker models 1.3.2 instead of the automorphic models 1.3.3 of cuspidal representations to define $Isob_1(F)$ for F global).

The reader will convince themselves that there is no natural application $Isob_1(F) \to Isob_1(F_v)$ if v is a place of the global field F.

1.4.3

Let us briefly indicate the construction of a new category $Isob_2$, equivalent to $Isob_1$, and where the tensor product is well-defined modulo the Main Conjecture. Its objects are the equivalence classes of isobaric representations: one can identify them with the data $\sigma_1 \boxplus \cdots \boxplus \sigma_r$ of $Isob_1$, taken modulo permutation.

Each object of $Isob_2$ can be identified with a formal sum $\oplus_\sigma n_\sigma \sigma$ (finite sum), where σ describes $\coprod_{n=1}^{\infty} \mathrm{Aut}^0(n)$. We define

$$\mathrm{Hom}_{\mathcal{G}}(\oplus n_\sigma \sigma, \oplus m_\sigma \sigma) = \bigoplus_\sigma \mathrm{Hom}(\mathbb{C}^{n_\sigma}, \mathbb{C}^{m_\sigma}).$$

One easily verifies that we have defined an abelian category, with the obvious composition of homomorphisms. There is an obvious map $Isob_1 \xrightarrow{\Phi} Isob_2$; using the fact, noted in 1.4.1, that $\mathrm{Hom}_G(\pi, \pi')$ for $\pi, \pi' \in Isob_1$ decomposes according to the isotypic components, we see that it is a functor, and that $\mathrm{Hom}_G(\Phi\pi, \Phi\pi') = \mathrm{Hom}_G(\pi, \pi')$ for $\pi, \pi' \in Isob_1$. The two categories are therefore equivalent (cf. e.g. [16, Prop. 1.19]).

One constructs analogously the category $Isob_2(F)$ for F local, from 1.4.2. One then notes that:

(1) Modulo the Main Conjecture, $\boxtimes$ is well-defined as a map $(Isob_2)^2 \to Isob_2$ (F global).

(2) If v is a place of a global field F, there is a restriction map $Isob_2(F) \to Isob_2(F_v)$.

1.4.4

The constructions of paragraphs 1.4.1 and 1.4.3 can be transposed into the category Alg, replacing of course the direct sum $\boxplus$ by the operation $\overset{T}{\boxplus}$. In particular, one thus equips suitable subcategories Alg_1 or Alg_2 of Alg with a structure of an abelian category.

By restriction, one thus has structures of abelian categories on all the categories of 1.2.

2 Some Well-Known Cases of the Principle of Functoriality

In this paragraph we review some cases where the functoriality principle has been established. We are mainly interested in groups related to $GL(n)$ and the examples described in 1.1.

Let us first recall the statement of the functoriality principle. Let H be a connected reductive group over F, ${}^L H = {}^L H^0 \times W_F$ the Weil form of its L-group. If $\pi = \otimes \pi_v$ is an automorphic representation of $H(\mathbb{A}_F)$, unramified outside a finite set of places S (containing the infinite places), one associates to it, for $v \notin S$, its Hecke matrix: it is a conjugacy class (under ${}^L H^0$) in ${}^L H$, mapping by the projection ${}^L H \to W_F$ to a Frobenius element for F_v.

Let then H_1, H_2 be two groups, $\varphi : {}^L H_1 \to {}^L H_2$ a homomorphism of L-groups [10 a]. If π_1 is an automorphic representation of $H_1(\mathbb{A}_F)$, the functoriality principle asserts the existence of an automorphic representation π_2 of $H_2(\mathbb{A})$ such that, at almost every place, $t_{\pi_2, v} = \varphi(t_{\pi_1, v})$. In the absence of "multiplicity one theorems" (cf. Thm. 1.1), it is not necessarily unique.

Let us recall some illustrations of the principle that have been proven for $GL(n)$ and some classical groups.

2.1 Automorphic Induction and Restriction

The homomorphisms of L-groups are described in 1.1. In the case of a cyclic extension, the following theorem was proven by Arthur and the author:

Theorem 2.1. *Let F_1/F be a cyclic extension of number fields of degree d. Then:*

(i) If π_F is a representation induced from cuspidal (category IC of 1.2.2) of $GL(n, \mathbb{A}_{F_1})$, the representation $\iota_{F_1}^F \pi_{F_1}$ of $GL(nd, \mathbb{A}_F)$ exists, and is uniquely determined; it belongs to IC. [1]

(ii) If π_F is a representation of $GL(n, \mathbb{A}_F)$ belonging to IC, the representation $\rho_{F/F_1}\pi_F$ of $GL(n, \mathbb{A}_{F_1})$ exists; it is uniquely determined and belongs to IC.

In the case where $n = 1$, part (i) is due to Kazhdan and Flicker [34, 25 a]: it is the "Hecke" functoriality sending Grössencharaktere on F_1 to representations of $GL(d, \mathbb{A}_F)$.

For a more detailed description of the results, and of the image of ρ_{F/F_1} in the cyclic case, we refer the reader to [18 d, e].

2.2 Symmetric Square

This is the functoriality between $GL(2)$ and $GL(3)$ deduced from the homomorphism of L-groups $GL(2, \mathbb{C}) \to GL(3, \mathbb{C})$ given by the symmetric square. It was proven by Gelbart and Jacquet [27]. Flicker, following earlier work by Jacquet and Langlands, announced a different proof, using the trace formula, which would allow to characterize its image.

2.3 Functoriality Associated to the Exterior Square

Let $GSp(4)$ be the group of symplectic similitudes on a space of dimension 4. It is a split group, whose L-group is $GSp(4, \mathbb{C}) \times W_F$. The "obvious" functoriality associated to the morphism $GSp(4, \mathbb{C}) \to GL(4, \mathbb{C})$ should have as image the representations π of $GL(4, \mathbb{C})$ such that the L-function $L(s, \pi, \Lambda_2)$, associated to the exterior square, has a pole. This was proven by Jacquet and Shalika (see Jacquet's talk in this volume, and [6 a] for a statement).

[1] [TRANS] Translator did not understand symbols, maybe π_{F_1}?

2.4 Unitary Groups

Let F_1/F be a quadratic extension of number fields. Let G be a unitary group over F, associated to an F_1/F -hermitian form on a space of dimension n over F_1. Its L-group is $GL(n,\mathbb{C}) \times W_F$, W_F operating via $Gal(F_1/F)$, whose non-trivial element σ acts on $GL(n,\mathbb{C})$ by the outer automorphism of order 2. We have $G \times_F F_1 \cong GL(n)/F_1$; base change should relate automorphic representations of $GL(n,\mathbb{A}_{F_1})$ fixed by the automorphism $g \mapsto^{\sigma} ({}^t g^{-1})$ and automorphic representations of $GL(\mathbb{A}_F)$.

In general, the problem is complicated by phenomena of L-indistinguishability: it is solved, for $n = 3$, by Rogawski [43 b].

Rapoport, Zink and Kottwitz discovered that L-indistinguishability was defeated for certain unitary groups coming from division algebras over F_1 equipped with an involution of the second kind (this case has also been studied, under certain local hypotheses and using an apparently unproven "Fundamental Lemma," by Flicker [25 b]). We shall make use of Kottwitz's results in §5.

3 Questions of Rationality

In this chapter, we systematically study the field of definition of automorphic representations; this is the generalization of the field generated by the values of a Hecke character, or by the Fourier coefficients of a modular form. We formulate a conjecture, according to which algebraic representations are characterized by their rationality properties. We show that regular cuspidal algebraic representations are defined over number fields, and that their conjugates under $\mathrm{Aut}(\mathbb{C})$ are automorphic. We deduce from this the generalization to $GL(n)$ of an Eichler-Shimura theorem (Thm. 3.19).

3.1 The Field of Rationality of an Automorphic Representation

The definitions in this paragraph are those of Waldspurger [58], and I have benefited from numerous discussions with him during the writing.

Let $\pi = \otimes \pi_v$ be an irreducible admissible representation of $GL(n,\mathbb{A})$. We denote by $\pi_f = \otimes_{v \nmid \infty} \pi_v$ its finite part: it is an irreducible admissible representation of $GL(n,\mathbb{A}_f)$, defined up to isomorphism by π.

Since $G(\mathbb{A}_f) = GL(n,\mathbb{A}_f)$ is a totally discontinuous group, we can consider its smooth representations over an arbitrary field (of characteristic zero).

For $\sigma \in \mathrm{Aut}(\mathbb{C})$, let ${}^{\sigma}\pi_f$ be the representation defined as follows: if V is the space of π_f, we set ${}^{\sigma}V = V \otimes_{\mathbb{C},\sigma^{-1}} \mathbb{C}$ (the tensor product of V with $\mathbb{C}$ over $\mathbb{C}$,

the homomorphism $\mathbb{C} \to \mathbb{C}$ being σ^{-1}); ${}^{\sigma}V$ is a $G(\mathbb{A}_f)$-module by the action on the first factor.

It is clear that ${}^{\sigma}\pi_f$ is again irreducible admissible; we have ${}^{\tau\sigma}\pi_f \cong {}^{\tau}({}^{\sigma}\pi_f)$. The field of rationality of the representation π_f is the subfield of $\mathbb{C}$ fixed by the automorphisms σ such that ${}^{\sigma}\pi_f \cong \pi_f$. We denote it by $Q(\pi_f)$.

If F is a non-archimedean local field, we similarly define the field of rationality $\mathbb{Q}(\pi)$ of an irreducible admissible representation π of $G(F)$. It is clear that the factor, at the finite place v, of ${}^{\sigma}\pi_f$, is isomorphic to ${}^{\sigma}\pi_v$. We deduce that the stabilizer of π_f in $\mathrm{Aut}(\mathbb{C})$ is the intersection of the stabilizers of the π_v.

In general, a representation, even of a finite group, is not defined over its field of rationality. For $GL(n)$, on the other hand, we have:

Proposition 3.1. *Let $(\pi, V)(\pi = \otimes\pi_v)$ be an irreducible admissible complex representation of $G(\mathbb{A}_f)$. Let $E = \mathbb{Q}(\pi)$.*

(i) There exists a vector subspace over E, V_E, of V such that $V_E \otimes_E \mathbb{C} = V$, and stable under $G(\mathbb{A}_f)$. In particular, π is obtained by extension of scalars from a representation over E.

(ii) The space V_E is unique up to complex homotheties.

(iii) The field $\mathbb{Q}(\pi)$ is the composite of the $\mathbb{Q}(\pi_v)$.

We also have the local analogue:

Proposition 3.2. *Let (π, V) be an irreducible admissible complex representation of $G(F)$ for F a non-archimedean local field, $E = \mathbb{Q}(\pi)$. Then properties (i), (ii) of Proposition 3.1 are true for (π, V).*

Let us first remark that Proposition 3.1 follows from the local assertion 3.2. If indeed π_v is obtained by extension of scalars from the representation $\pi_v^{E_v}$ (where $E_v = \mathbb{Q}(\pi_v)$), we construct the tensor product $\otimes_v \pi_v$, a representation of $G(\mathbb{A}_f)$ obviously defined over the composite E' of the $\mathbb{Q}(\pi_v)$. Extending scalars to $\mathbb{C}$, we obtain π; π can therefore be defined over E'; moreover, $E' \subset \mathbb{Q}(\pi)$ since an automorphism of C fixing π fixes the $\mathbb{Q}(\pi_v)$, hence E'. Therefore π can be defined over $\mathbb{Q}(\pi)$ and $E' = \mathbb{Q}(\pi)$ whence (i) and (iii). Assertion (ii) is demonstrated as follows [58, Lemma I.1]: Let V_1, V_2 be two E-vector subspaces of V such that $V_i \otimes_E \mathbb{C} = V$. If $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$, denote by σ_1, σ_2 the σ-linear automorphisms of V deduced from V_1, V_2.

The automorphism $\lambda_\sigma = \sigma_1^{-1}\sigma_2$ of V is then linear, and it commutes with the action of $G(\mathbb{A}_f)$. By Schur's lemma (which is true for irreducible admissible representations), λ_σ is scalar: we identify it with a complex number. Let then $0 \neq f \in V_2$. We can write $f = \sum e_i \otimes \lambda_i$, where (e_i) is a basis of V_1 over E and $\lambda_i \in \mathbb{C}$.

We then have

$$\sigma_2 f = f = \sum e_i \otimes \lambda_i,$$

$$\sigma_2 f = \lambda_\sigma \sigma_1 f = \lambda_\sigma \sum e_i \otimes \sigma(\lambda_i).$$

We deduce that $\lambda_\sigma = \lambda_i \sigma(\lambda_i)^{-1}$ for some i such that $\lambda_i \neq 0$, and for all σ. This implies that V_1 and V_2 are homothetic with ratio λ_i.

It remains to prove Proposition 3.2[2]. Suppose first that π is generic. Then, according to Jacquet, Piatetski-Shapiro and Shalika [32 b], there exists an open compact subgroup $K \subset G(F)$, such that the trivial representation of K appears with multiplicity 1 in π. The Proposition then follows from Waldspurger's Lemma I.1 [58].

In the general case, we will reduce to the generic case using the classification of Zelevinsky. As this will be useful to us later, we prefer to use the Langlands classification reformulated in the framework of Zelevinsky's theory [58] by Rodier [42].

If τ is a supercuspidal representation of $GL(a, F_v)$ and $b = [b', \ldots, b'']$ is a sequence of consecutive integers of length $|b| = b'' - b' + 1$, let $L(b, \tau)$ be the unique irreducible quotient of

$$\mathrm{Ind}_{P(F)}^{GL(a|b|,F)}(\tau|\ |^{b'} \otimes \cdots \otimes \tau|\ |^{b''}), \tag{3.1}$$

where P is the homogeneous parabolic of type $a, \ldots, a$. A sequence of supercuspidal representations of the form $(\tau|\ |^{b'}, \ldots, \otimes \tau|\ |^{b''})$ is a segment. We denote it by $\Delta(b, \tau)$. We write $\Delta_1 \to \Delta_2$ (Δ_1 precedes Δ_2) if $\Delta_1 \neq \Delta_2$, $b_1' \leq b_2'$ and $b_1'' \leq b_2''$. Note that $L(b, \tau)$ is square-integrable.

We then know that every irreducible representation π of $GL(n, F)$ is written as a full induced representation

$$\mathrm{Ind}_{Q(F)}^{G(F)}(\pi_1 \otimes \cdots \otimes \pi_r) \tag{3.2}$$

where Q is the parabolic of type $(n_1, \ldots, n_r)$ and the π_i are of the following type: π_i is the unique irreducible quotient of

$$\mathrm{Ind}_{P_i(F)}^{G_{n_i}(F)}(L(b_1^i, \tau_i) \otimes \cdots \otimes L(b_{s_i}^i, \tau_i)), \tag{3.3}$$

τ_i being a supercuspidal representation of $GL(a_i, F)$ and the segments $\Delta(b_j^i, \tau_i)$ satisfying the condition

$$\Delta(b_j^i, \tau_i) \to \Delta(b_k^i, \tau_i) \Rightarrow j > k. \tag{3.4}$$

[2]Waldspurger indicated to me a different proof of the Proposition, using directly degenerate Whittaker models.

Moreover, the τ_i are then uniquely determined if we assume them distinct modulo torsion by integer powers of $|\det|$. The representation (3.2) is unique up to permutation of the τ_i or the π_i.

For the moment, the classification is not given in rational terms because unitary induction does not respect the rationality of representations. If P is a parabolic of G, of type $(n_1, \dots, n_r)$, we will verify later (Lemma 3.9) that the induction operation used in Definition 1.9, which associates to $(\pi_1, \dots, \pi_r)$

$$\mathrm{Ind}_{P(F)}^{G(F)}(\pi_1|\ |^{\frac{1-n_1}{2}} \otimes \cdots \otimes \pi_r|\ |^{\frac{1-n_r}{2}})\ |\ |^{\frac{n-1}{2}},$$

which we will denote by $\mathrm{Ind}_{P(F)}^{G(F)}(\pi_1 \otimes \cdots \otimes \pi_r)$, preserves the field of definition.

We therefore reformulate the described classification by replacing at each step (3.1, 3.2, 3.3) the unitary induction by the twisted induction. The obtained classification, called **rational classification**, has the properties mentioned previously.

Suppose then that π is associated, by the rational classification, to a family of segments $\Delta(b_j^i, \tau_i)$ satisfying (3.4). By uniqueness, the group $\Sigma = \mathrm{Aut}(\mathbb{C}, E)$, where $E = \mathbb{Q}(\pi)$, must permute the τ_i. Moreover, if i, k belong to the same orbit I of Σ, we must have $s_i = s_k = s$ and $b_j^i = b_j^k$ for all $j = 1, \dots, s$. Consider then, for every j, the representation

$$\mathrm{Ind}_{R(F)}^{G_{ta|b|}}(\mathcal{L}(b, \tau_{i_1}) \otimes \cdots \otimes \mathcal{L}(b, \tau_{i_t}))$$

(twisted induction), where $I = \{i_1, \dots, i_t\}, b = b_j, \tau_i$ is a representation of $GL(a)(i \in I)$, and R is the obvious parabolic; we denote by $\mathcal{L}$ the Langlands quotient (3.1) for rational induction. This representation is irreducible and generic [58]. Since Σ permutes the τ_i, it is fixed by Σ and therefore, by the already proven result, can be defined over E. By considering these representations for all orbits and all segments, we then deduce, by induction by stages, that π can be realized as the unique irreducible quotient of an induced (rational!) representation defined over E.

It remains to prove that the quotient is defined over E. This is the subject of the following well-known lemma:

Lemma 3.2.1. Let $X = X_E \otimes_E \mathbb{C}$ be a vector space defined over E, $W \subset X$ a complex subspace of X stable by $\Sigma = \mathrm{Aut}(\mathbb{C}/E)$. Then $W = W_E \otimes_E \mathbb{C}$, where $W_E = W \cap X_E$.

We apply the lemma to the kernel W of the map $X \to V$, where X is the space of the induced representation, defined over E, of which V is the unique irreducible quotient. It is clear that W is stable by Σ, and this proves Proposition 3.2.

Proof of the Lemma: Let (e_i) be a basis of X_E over E. Let $v = e_1\lambda_1 + \cdots + e_n\lambda_n \in W$, and suppose that every vector of W admitting such an expression of length less than or equal to n belongs to $W_E \otimes \mathbb{C}$. We can therefore assume

$\lambda_i \neq 0$ for all i; let moreover $\lambda_1 = 1$. If $\lambda_i \notin E$, there exists $\sigma \in \Sigma$ such that $\sigma\lambda_i \neq \lambda_i$. Consider then

$$v' = \sigma v - v = e_2(\sigma\lambda_2 - \lambda_2) + \cdots + e_n(\sigma\lambda_n - \lambda_n).$$

It belongs to W; by the induction hypothesis, we therefore have $v' \in W_E \otimes \mathbb{C}$. Let then $\xi_i = \sigma\lambda_i - \lambda_i \neq 0$. Then $v'' = v\xi_i - v'\lambda_i$ has a zero component along e_i; therefore, by induction, $v'' \in W_E \otimes \mathbb{C}$: we finally have $v \in W_E \otimes \mathbb{C}$.

3.2 Unramified Representations: Rational Parametrization

The Langlands parametrization of unramified representations is not rational. We must therefore transform it by a Tate twist.

In this paragraph, F is a non-archimedean local field. We write G for $G(F)$, K for $G(\mathcal{O}_F)$. Let $T \subset G$ be the diagonal torus, $K_T = T(\mathcal{O})$. The Hecke algebras $\mathcal{H}(G, K)$ and $\mathcal{H}(T, K_T)$[3] are defined over $\mathbb{Q}$ (we write $\mathcal{H} = \mathcal{H}(G, K)$, $\mathcal{H}_T = \mathcal{H}(T, K_T)$).

3.2.1 Satake transform.

Recall that the Satake transform of $f \in \mathcal{H}$ is defined by $Sf(h) = \delta_{P_0}(h)^{\frac{1}{2}} \int_{N_0} f(hn)dn, h \in T$ (notations 1.1; δ_{P_0} is the modulus of $P_0 = TN_0$).

Definition 3.3. If $f \in \mathcal{H}$, we define

$$S^T f(h) = |\det h|^{\frac{n-1}{2}} Sf(h) \quad (h \in T).$$

We verify that S^T is an isomorphism defined over $\mathbb{Q}$ from $\mathcal{H}$ to $\mathcal{H}_T^W$. We identify $\mathcal{H}_T(\mathbb{Q})$ with $\mathbb{Q}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ by sending the characteristic function of $\bar{\omega}^\lambda$ to z^λ ($\lambda \in \mathbb{Z}$), where $\bar{\omega}^\lambda$ is the diagonal matrix $(\bar{\omega}^{\lambda_1}, \ldots, \bar{\omega}^{\lambda_n})$. The dual torus ${}^L T^0$ is thus canonically endowed with a rational structure (we will denote by $\hat{f}^T$, for $f \in \mathcal{H}$, the element of $\mathbb{C}({}^L T^0)^W$ thus defined).

3.2.2 Unramified representations.

Definition 3.4. If $\chi = (\chi_1, \ldots, \chi_n)$, where the χ_i are unramified characters of $F^\times$, we set

$$\pi^T(\chi_1, \ldots, \chi_n) = \pi(\chi_1 |\ |^{\frac{n-1}{2}}, \ldots, \chi_n |\ |^{\frac{n-1}{2}}).$$

On the right-hand side, $\pi()$ denotes the Langlands subquotient of the induced representation; it is unramified. We could write, using the local analogue of Definition 1.10:

$$\pi^T(\chi_1, \ldots, \chi_n) = \chi_1 \overset{T}{\boxplus} \chi_2 \overset{T}{\boxplus} \ldots \overset{T}{\boxplus} \chi_n.$$

[3] [TRANS] Original paper has typo, I think

We then verify:

Lemma 3.5. *(i) If $f \in \mathcal{H}$, f acts on the spherical vector of $\pi^T(\chi)$ by $\hat{f}^T(t_\chi)$, where $t_\chi = (\chi_1(\bar{\omega}), \dots, \chi_n(\bar{\omega})) \in {}^L T^0$.*

(ii) If $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have ${}^\sigma(\pi^T(\chi)) \cong \pi^T({}^\sigma\chi)$.

(iii) In particular, the field of rationality of $\pi^T(\chi)$ is that of the orbit of $t_\chi \in {}^L T^0(\mathbb{C})$ under W.

Part (i) is obvious. Part (ii) follows since the map $f \mapsto \hat{f}^T$ is defined over Q, and the unramified representations are classified by the characters of the Hecke algebra. As these correspond to ${}^L T^0/W$, (iii) follows from (ii).

Remark. If π is an unramified representation of $G(F)$, we will denote by t_π^T its Hecke matrix for the parametrization given by Definition 3.4; this is t_π multiplied by the diagonal matrix $q_v^{\frac{1-n}{2}}$.

3.3 The Action of $\mathrm{Aut}(\mathbb{C})$ on the Archimedean Type of Algebraic Representations

We will begin by reformulating the construction used in paragraph 1.2.3 to define the category Alg. Denote by I the set of embeddings $\iota : F \to \mathbb{C}$. The set of infinite places of F is thus identified with I modulo the action of complex conjugation.

To any representation $\pi \in Alg(n)$, we associate its type at infinity, an element of $\mathbb{Z}^n/\mathfrak{S}_n)^I$, as follows: let $\pi' = \pi|\,|^{\frac{1-n}{2}}$; the Langlands parametrization associates to it, by hypothesis, algebraic characters of $\mathbb{C}^\times$.

If ι is a real embedding, π'_ι is associated to a representation of $W_\mathbb{C} = \mathbb{C}^\times$ of the form $(z^{p_1}\bar{z}^{p_2}, z^{p_2}\bar{z}^{p_1}, \dots, (z\bar{z})^{p'_{2r+1}}, \dots, (z\bar{z})^{p'_n})$ (cf. after Definition 1.8). We set $p_\iota = (p_1, p_2, \dots, p_{2r}, p'_{2r+1}, \dots, p'_n)$ considered as an element of $\mathbb{Z}^n/\mathfrak{S}_n$.

If ι is a complex embedding, it defines an identification $F_v = \mathbb{C}$ where v is the place deduced from ι. We have $W_\mathbb{C} = \mathbb{C}^\times$ (canonically), and π'_v is the representation of $GL(n, \mathbb{C})$ associated to the characters $(z^{p_1}\bar{z}^{q_1}, \dots, z^{p_n}\bar{z}^{q_n})$. We set $p_\iota = (p_1, \dots, p_n)$ (modulo $\mathfrak{S}_n$). It is clear that then $p_{\bar{\iota}} = (q_1, \dots, q_n)$ where $\bar{\iota}$ denotes complex conjugation.

Definition 3.6. Let $\pi \in Alg(n)$, and let $p(\pi) = p = (p_\iota)_{\iota \in I}$ be its type at infinity. If $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define ${}^\sigma p$ by $({}^\sigma p)_\iota = p_{\sigma^{-1}\iota}$ $(\iota \in I)$.

In the case of $Alg(1)$ - that is to say, of algebraic Hecke characters - this amounts to the usual definition of the action of $\mathrm{Aut}(\mathbb{C})$ on types: if $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is an algebraic Hecke character, we have, up to an Artin character, $\chi_\iota(x) = x^{p_\iota}$ (ι real) and $\chi_v(z) = z^{p_\iota}\bar{z}^{q_\iota}$ (ι complex, F_v identified with $\mathbb{C}$ via ι). If $\sigma \in \mathrm{Aut}(\mathbb{C})$,

we verify that the type of ${}^{\sigma}\chi$ (σ acting on the values of χ_f on $\mathbb{A}_f^{\times}$) is indeed that prescribed by Definition 3.2, using the formula $\chi_\infty(x)\chi_f(x) = 1$ ($x \in F^{\times}$).

Remark. The type at infinity of π - which one should rather call its algebraic type at infinity - does not determine π_∞; this is already the case for $n = 1$: for a Grössencharakter, the algebraic type only defines χ_∞ on $(F_\infty^{\times})^+$. Nevertheless, for π **cuspidal**, the Purity Lemma 4.9 shows that $p(\pi)$ determines the character of π on the set of elements of $G(F_\infty)$ which are squares: cf. Corollary 4.11. This is false if π is not cuspidal.

3.4 Conjectures and Verifications

3.4.1 Conjectures

Conjecture 3.7. *Let $\pi \in Alg(n)$, $p = (p_\iota)_{\iota \in I}$ its type at infinity. Then*

(i) π_f is defined over a subfield $E \subset \mathbb{C}$ which is a number field.

(ii) For all $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^{\sigma}\pi_f$ is the finite part of a representation ${}^{\sigma}\pi \in Alg(n)$ whose type at infinity is ${}^{\sigma}p$.

Conjecture 3.7 could be formulated using the L-function of π; it implies that $L_f(\pi, s + \frac{1-n}{2})$ is given by a Dirichlet series with coefficients in E (cf. Lemma 4.7) (L_f is the non-archimedean part of the L-function; the function $L(\pi)$ is that of Langlands: functional equation $s \mapsto 1 - s$, exchanging π and $\tilde{\pi}$). It implies, in particular, that at unramified places,

$$t_{\pi,v}^T \in ({}^L T^0/W)(E). \tag{3.5}$$

For the subcategory of Artin representations, we are led to further conjecture:
Conjecture 3.7' If $\pi \in Art(n)$, the eigenvalues of $t_{\pi,v}^T$ (v unramified) are roots of unity.

As Weil had done for algebraic Hecke characters, it seems natural to formulate a reciprocal conjecture, which is a question of transcendence theory. In the strongest form:

Conjecture 3.8. *Let $\pi \in Aut(n)$, and suppose that at all places unramified for π, except for a finite number:*

$$t_{\pi,v}^T \in ({}^L T^0/W)(E), \quad E \subset \mathbb{C}$$

where E is a number field. Then $\pi \in Alg(n)$.

We now verify the compatibility of the conjecture with some functorial operations.

3.4.2 Central characters.

If $\pi \in Alg(n)$, its central character ω_π is an algebraic Hecke character. In particular, the obvious consequences of Conjecture 3.7 for central characters are true (recall that Conjecture 3.8 for Aut (1) has been proved by Waldschmidt [57]).

3.4.3 Parabolic induction.

The conjecture is compatible with the operation $\overset{T}{\boxplus}$:

Lemma 3.9. *Let $\pi = \pi' \overset{T}{\boxplus} \pi''$, $\pi', \pi'' \in Alg$.*

(i) If π'_f, π''_f are defined over E, π is also.

(ii) Let p, p', p'' be the types at infinity of π, π', π''. Then $p = p' \oplus p''$ in an obvious sense.

Assertion (ii) is obvious. For (i), one must first verify that the (re)-normalized induction associating to π'_v and π''_v the representation

$$\rho_v = \mathrm{Ind}_{P(F_v)}^{G(F_v)}(\pi'_v|\ |^{\frac{1-n'}{2}} \otimes \pi''_v|\ |^{\frac{1-n''}{2}} \otimes 1) \otimes |\ |^{\frac{n'+n''-1}{2}}$$

is defined over $\mathbb{Q}$, i.e., it preserves the field of rationality of the representations. The parasitic module, which introduces quadratic irrationals in the definition of the unitary induction Ind_P^G, is given by

$$|\det m'|^{\frac{n''}{2}} |\det m''|^{\frac{-n'}{2}} = \delta_P^{\frac{1}{2}}(m), \quad m = \begin{pmatrix} m' & 0 \\ 0 & m'' \end{pmatrix} \in M.$$

The indicated twists replace it by $|\det m'|^{n''}$, a rational number.

One must then verify that the Langlands subquotient of ρ_v is defined by rational conditions (this is not clear from the Langlands definition, which is expressed by a positivity condition). For this, we use the rational Langlands classification described in the proof of Proposition 3.2. It realizes any representation π of $GL(n, F_v)$ as an induced representation

$$\mathrm{Ind}_{Q(F)}^{G(F)}(\pi_1 \otimes \cdots \otimes \pi_r) \tag{3.6}$$

(notations 3.1: Ind denotes rational induction) where π_i is the unique irreducible quotient of

$$\rho_i = \mathcal{I}\mathrm{nd}_{P_i(F)}^{G_{n_i}(F)}(\mathcal{L}(b_1^i, \tau_i) \otimes \cdots \otimes \mathcal{L}(b_s^i, \tau_i)) \tag{3.7}$$

(recall that $\mathcal{L}$ is the Langlands quotient (3.1) for rational induction). The generic representation of which π is the Langlands quotient (and uniquely defined by this condition) is then

$$\rho = \mathcal{I}\mathrm{nd}_{Q(F)}^{G(F)}(\rho_1 \otimes \cdots \otimes \rho_r)$$

It is clear, since rational induction commutes with the action of Aut $(\mathbb{C})$, that the Langlands quotient of ${}^{\sigma}\rho_v$ is then ${}^{\sigma}\pi_v$, if π_v is that of ρ_v. In particular, the field of definition of π_v is contained in that of ρ_v, whence Lemma 3.9.

Remark. (a) The reader might worry that we used Lemma 3.9 to prove Proposition 3.2. We had obviously only used the rationality property of the full induced representation ρ.

(b) The field of rationality of $\pi' \overset{T}{\boxplus} \pi''$ can of course be smaller than that of π' and π'': think for example of the case of two Artin characters $\chi' : \mathbb{A}_f^{\times} \to E^{\times}$, where $E/\mathbb{Q}$ is quadratic, and $\chi'' = (\chi')^{\sigma}$ where σ generates $\mathrm{Gal}(E/\mathbb{Q})$.

(c) In the case where π'_v and π''_v are **unramified**, we know that the Langlands subquotient is the unique unramified subquotient of the induced representation: it is clear that this is a rational condition, and this gives another proof of (i) at these places. One would like to know that, in all cases, the Langlands subquotient is defined by conditions of ramification (K-minimal type). This could result from recent work of Howe, Moy and Bushnell.

Lemma 3.9 reduces the verification of Conjecture 3.7 to the case of cuspidal representations (cf. Lemma 1.12).

3.4.4 Base change.

Let K/F be an extension of global fields.

Let first $\pi \in Alg^0(n, F)$ be a cuspidal representation. Let $\Pi \in Alg(n, K)$ be the representation of $GL(n, \mathbb{A}_K)$ deduced (conjecturally) from Π by base change (§1.1, 2). One then verifies easily that, at unramified places, equation (3.1) for π implies equation (3.1) for Π. In particular, if π_v is defined over $E \subset \mathbb{C}$, Π_w is $(w|v)$ at the unramified places of π.

Let ι be an embedding $F \to \overline{\mathbb{Q}}$, η an embedding of K extending ι. The conjectural properties of base change at archimedean places imply that $p(\Pi, \eta) = p(\pi, \iota)$ (notations 3.3). Since $\sigma^{-1}\eta$ extends $\sigma^{-1}\iota$ ($\sigma \in \mathrm{Aut}(\mathbb{C})$), we see therefore that Conjecture 3.4 for π implies it for Π (at least at the unramified places of π, or archimedean ones). In the solvable case, we have more; note that Π exists by the results of [2].

Lemma 3.10 (K/F solvable)**.** *Let $\pi \in Alg^0(n,F)$, $\Pi \in Alg(n,K)$ the representation deduced from π by base change. Then Conjecture 3.4 for π implies it for Π.*

Proof. We reduce to K/F abelian (at intermediate stages, we no longer have $\pi \in Alg^0$, but $\pi \in Alg \cap IC$; we leave the ancillary arguments to the reader). It remains to consider the ramified places. In this case, we have a character identity [2, ch. I], when $w|v$ are finite places:

$$\text{trace}(\Pi_w(g)I_\sigma) = \text{trace } \pi_v(\mathcal{N}g),$$

$g \in GL(n,E)$ of regular norm, σ a generator of $\text{Gal}(K/F)$. We deduce that the twisted character of Π takes its values in E. Since the values of the twisted character on regular elements determine the representation [2, 18 b], this implies that Π is rational, and therefore defined, over E.

□

We have analogous results in the case of automorphic induction. Let $\pi \in Alg^0(n,K), \pi_F \in Alg(nd,F)$ conjecturally deduced from π (§1.1) by automorphic induction. One then verifies that at unramified places, $(\pi_F)_v$ is defined over E if π_w is for all $w|v$. At archimedean places, we have

$$p(\pi_F, \iota) = \oplus_\eta \, p(\pi, \eta)$$

where the sum is over embeddings $K \to \overline{\mathbb{Q}}$ extending $\iota : F \to \overline{\mathbb{Q}}$. We again deduce compatibility with Conjecture 3.4 at unramified or archimedean places. In the solvable case, one could prove the analogue of Lemma 3.10 modulo the following conjecture, which I have not been able to prove:

Conjecture 3.11. *Let K/F be a cyclic extension of local fields, of prime degree ℓ. Let σ be a generator of Gal(K/F). Let π be a supercuspidal representation of $GL(n,K)$ such that $\pi \not\cong \pi \circ \sigma$. Let π_F be the unique representation of $GL(n\ell,F)$ automorphically induced from π ([2, Prop. 6.6] ; [18 d]). Then, if π is defined over $E \subset \mathbb{C}$, π_F is defined over E.*

(we know that the restriction of π_F to the kernel of det: $GL(n\ell,F) \to F^\times/N_{K/F}(K^\times)$ is defined over E). [4]

3.5 The Case of Regular Representations

Definition 3.12. Let $\pi \in Alg(n), p = (p_\iota)_{\iota \in I}$ its type at infinity. We say that π is regular if, for all $\iota \in I, p_\iota = (p_1, \ldots, p_n)$ with $p_i \neq p_j (i \neq j)$.

[4] Added in proofs: As G. Henniart pointed out to me, Conjecture 3.11 follows easily from the fact that π_F is the unique representation of $G_{n\ell}(F)$ whose lift to K is equal to $\pi \boxplus (\pi \circ \sigma) \boxplus \cdots \boxplus \pi \circ \sigma^{\ell-1}$.

This amounts to assuming its infinitesimal character is **regular** (cf. the Remark after Definition 1.8).

In this paragraph, we verify Conjecture 3.7 for regular representations:

Theorem 3.13. *Let $\pi \in Alg(n)$, p its type at infinity. Suppose π is regular. Then π_f is defined over a number field $E \subset \mathbb{C}$; for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^{\sigma}\pi_f$ is the finite part of a representation ${}^{\sigma}\pi \in Alg^0(n)$, of type at infinity ${}^{\sigma}\pi$.*[5]

Remark. The representation ${}^{\sigma}\pi$ is then unique (Thm. 1.1).

Proof. The fact that π_f is defined over a number field results from the relations between automorphic forms and the cohomology of arithmetic quotients of $G(F_\infty)/K_\infty$ (for $GL(2)$ over $\mathbb{Q}$, this is the Eichler-Shimura isomorphism). We briefly describe the argument: cf. Harder [30 b] and Walspurger [58] for $GL(2)$. We will use the following result, which will be demonstrated in paragraph 4.4. □

PURITY LEMMA 4.9. Let $\pi \in Alg^0(n)$. - Then there exists $w \in \mathbb{Z}$ such that, for every infinite place v, the algebraic characters of $\mathbb{C}^\times$ associated to $\pi_v | \ |_v^{\frac{n-1}{2}}$ are of the form $z^p(\bar{z})^q$ with $p+q=w$.

In other words, $\pi | \ |^{\frac{n-1-w}{2}}$ is tempered at infinite places.

Let $L = \mathrm{Res}_{F/\mathbb{Q}} GL(n)$. We first associate to the type at infinity of π a rational representation, defined over $\overline{\mathbb{Q}}$, of L. We denote it by τ, and V its space. We have $\tau = \otimes_{\iota \in I} \tau_\iota$.

If ι is a complex embedding, let $\bar{\iota}$ be its conjugate and v the associated place. Identify F_v with $\mathbb{C}$ via ι (cf. §3.3); let $\pi_v = \pi_{\mathbb{C}}(z^{a_1}\bar{z}^{b_1}, \dots, z^{a_n}\bar{z}^{b_n})$ be the expression of π_v in the Langlands classification; we have

$$a_i = p_i + \frac{n-1}{2}, \quad b_i = q_i + \frac{n-1}{2}$$

$p_i, q_i \in \mathbb{Z}$ and $(p_i) = p_\iota, (q_i) = p_{\bar{\iota}}$. We can order the indices so that

$$a_1 > a_2 > \cdots > a_n$$

whence, by the Purity Lemma:

$$b_1 < b_2 < \cdots < b_n.$$

We set

$$m_\iota = (p_1, p_2 + 1, p_3 + 2, \dots, p_n + (n-1)) \tag{3.8}$$

i.e.,

$$m_\iota = a - \delta \in \mathbb{C}^n, \quad a = (a_1, \dots, a_n)$$

[5] [TRANS] Maybe typo: ${}^{\sigma}p$

where δ is the half-sum of roots of $\mathfrak{gl}(n,C)$ in $\mathfrak{h}^*$ (cf. Remark after Definition 1.8):

$$\delta = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}\right) \in \mathbb{C}^n.$$

The definition applied to $\bar{\iota}$ gives

$$m_{\bar{\iota}} = (q_n, q_{n-1}+1, \ldots, q_1+(n-1));$$

m_ι and $m_{\bar{\iota}}$ identify with dominant weights for $GL(n)$ relative to the diagonal torus, because they are sequences of integers $(x_1, \ldots, x_n)$ with $x_1 \geq x_2 \geq \cdots \geq x_n$. We define τ_ι as the rational representation of highest weight m_ι relative to the root system on $\mathfrak{h} \cong \mathbb{C}^n$ associated to the standard minimal parabolic.

The choice of ι gives an isomorphism from $GL(n,F_v)$ to $GL(n,\mathbb{C})$, whence $GL(n,F_v) \times_{\mathbb{R}} \mathbb{C} \cong GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$. For this identification, the infinitesimal character of the representation $\tau_\iota \otimes \tau_{\bar{\iota}}$ identifies with

$$(m_\iota + \delta, m_{\bar{\iota}} + \delta) \in \mathbb{C}^n \times \mathbb{C}^n.$$

We verify that $m_{\bar{\iota}} + \delta = w_0 b$, where $w_0 \in \mathfrak{S}_n$ is the element of greatest length for the choice of roots; modulo the complex Weyl group, the infinitesimal character of $\tau_\iota \otimes \tau_{\bar{\iota}}$ is therefore equal to (a,b), that is, to that of π_v.

Let ι be a real embedding, and v the associated place. If $p_\iota = (p_1, \ldots, p_n)$, where we assume $p_1 > p_2 > \cdots > p_n$ we again define m_ι by formula (3.5), and τ_ι as before. Well-known calculations (cf. e.g. [51 b]; [18 c, p. 482]) show again that π_v has the same infinitesimal character as $\tau_v = \tau_\iota$. We finally define $\tau = \otimes_{\iota \in I} \tau_\iota$. Note that the central character of τ - an algebraic character of $\mathrm{Res}_{F/\mathbb{Q}} GL(1)$ - is equal to the type at infinity of the central character of π.

If v is an infinite place, let $\mathfrak{g}_v$ be the complex Lie algebra of $GL(n,F_v)$.

Let A_v be the topological identity component of the maximal real split torus of $GL(n,F_v)$ (considered as a real group), $\mathfrak{a}_v = \mathrm{Lie}(A_v) \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathfrak{g}_v = \mathfrak{a}_v \times \tilde{\mathfrak{g}}_v$ be the Langlands decomposition of $\mathfrak{g}_v$: $\tilde{\mathfrak{g}}_v$ is thus the set of matrices of $\mathfrak{g}_v$ whose trace is imaginary, and is invariant under the standard maximal compact subgroup K_v, equal to $O(n)$ or U_n. We denote by $H^\bullet(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \tau_v)$ the $(\tilde{\mathfrak{g}}_v, K_v)$-cohomology spaces [14]. If $\mathfrak{a}$ is a vector space, we denote by $\Lambda^\bullet \mathfrak{a}$ the graded vector space equal to $\Lambda^i \mathfrak{a}$ $(0 \leq i \leq \dim \mathfrak{a})$, to 0 elsewhere; if $\mathfrak{a} = \{0\}$, $\Lambda^\bullet \mathfrak{a} = \mathbb{C}$ in dimension 0.

Lemma 3.14. *Let v be an infinite place. Then there exists a character of order 2, ϵ_v, of $F_v^\times$ such that, denoting by $\pi_v \otimes \epsilon_v$ the representation π_v twisted by $\epsilon_v \circ \det$:*

$$\begin{aligned} H^\bullet(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \epsilon_v \otimes \tilde{\tau}_v) &\cong \Lambda^{\bullet - \frac{n-1}{2}} \mathbb{C}^{n-1} \quad (v \textit{ complex}) \\ &\cong \Lambda^{\bullet - m^2} \mathbb{C}^{m-1} \quad (v \textit{ real, } n = 2m) \\ &\cong \Lambda^{\bullet - m(m+1)} \mathbb{C}^m \quad (v \textit{ real, } n = 2m+1). \end{aligned}$$

Proof. By Lemma 4.9, the restriction of π_v to the group $\tilde{G}_v = \{g \in G_v : |\det g| = 1\}$ is tempered. Consider first the case where $F_v \cong \mathbb{C}$. In this case, $\epsilon_v = 1$. The representation τ_v of $GL(n, F_v)$ has for complexification $\tau_\iota \otimes \tau_{\bar{\iota}}$, representation of $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$. □

The representations τ_v and π_v have the same central character and the same infinitesimal character; $\pi_v|_{\tilde{G}_v}$ being tempered, Lemma 3.4 follows easily from the standard calculation of the cohomology of complex groups [24].

Consider now F_v real, and suppose first $n = 2m$ even. Let r_v be the Langlands parameter of π_v restricted to $\mathbb{C}^\times$; it is written

$$(z^{a_1}(\bar{z})^{a_2}, z^{a_2}\bar{z}^{a_1}, \dots, z^{a_{2r}}\bar{z}^{a_{2r-1}}, (z\bar{z})^{a_{2r+1}}, \dots, (z\bar{z})^{a_n})$$

with $a_i = p_i + \frac{n-1}{2}, p_i \in \mathbb{Z}$ $a_1 \neq a_2, \dots, a_{2r} \neq a_{2r-1}$. By regularity, we actually have $a_i \neq a_j$ $\forall i \neq j$. By the Purity Lemma, we moreover have

$$w - n - 1 = a_1 + a_2 = a_3 + a_4 = \cdots = 2a_{r+1} = \cdots = 2a_n.$$

There can therefore only be one term in $(z\bar{z})^{a_j}$; n being even, there is none; r_v is therefore written

$$(z^{a_1}\bar{z}^{a_2}, z^{a_2}\bar{z}^{a_1}, \dots, z^{a_{2m}}\bar{z}^{a_{2m-1}}).$$

We will assume that $a_1 < a_2$, $a_3 < a_4$, …, $a_{2m-1} < a_{2m}$. We can further assume $a_1 < a_3 < \cdots < a_{2m-1}$, whence from the complementarity relations due to the Purity Lemma[6]:

$$a_2 > a_4 > \cdots > a_{2m} > a_{2m-1} > \cdots > a_3 > a_1.$$

We see therefore that the parameters $p = p_\iota$, $m = m_\iota$ associated to ω_v are written

$$p = (a_2 - \frac{n-1}{2}, a_4 - \frac{n-1}{2}, \dots, a_1 - \frac{n-1}{2})$$

$$m = (a_2 - \frac{n-1}{2}, a_4 - \frac{n-3}{2}, \dots, a_1 + \frac{n-1}{2}).$$

Consider first the case $n = 2$. The representation π_v of the discrete series of $GL(2, \mathbb{R})$ associated by the Langlands classification to $(z^{a_1}\bar{z}^{a_2}, z^{a_2}\bar{z}^{a_1})$ fits into a non-split exact sequence:

$$0 \to \pi_v \to \mathrm{Ind}(\nu_2, \nu_1) \to F(\nu_2, \nu_1) \to 0$$

where the central representation is induced, from the Borel subgroup, from the characters

$$\nu_2(x) = |x|^{a_2}(\mathrm{sgn}\ x)^{\epsilon_2}$$

[6] Attention: this notation is not consistent with that used before (3.5): the order of the a_i is different.

$$\nu_1(x) = |x|^{a_1}(\operatorname{sgn} x)^{\epsilon_1}$$

with

$$\epsilon_1 + \epsilon_2 \equiv a_1 - a_2 - 1 \pmod 2,$$

ϵ_1 and ϵ_2 being arbitrary subject to this condition (recall that $a_1 < a_2$). The quotient $F(\nu_2, \nu_1)$, of finite dimension, has highest weight

$$\begin{pmatrix} x & \\ & y \end{pmatrix} \mapsto \nu_2(x)|x|^{-\frac{1}{2}}\nu_1(y)|y|^{\frac{1}{2}}.$$

Since $a_i = p_i + \frac{1}{2}$, $F(\nu_2, \nu_1)$ is rational if $\epsilon_1 \equiv p_1 + 1 \pmod 2$, $\epsilon_2 \equiv p_2 \pmod 2$, which is compatible with the relations between the ϵ; this is the representation $\tau_v = \tau_\iota$ defined previously. The exact sequence above shows that $\operatorname{Ext}^1_{(\tilde{\mathfrak{g}}_v, K_v)}(\pi_v, \tau_v) = H^1(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \tilde{\tau}_v)$ is non-zero; it is well known that in this case

$$H^\bullet(\tilde{\mathfrak{g}}_v, K_v; \pi_v \otimes \tilde{\tau}_v) \cong \Lambda^{\bullet-1}\mathbb{C}^0.$$

Let us now move to the general case where n is even; let $n = 2m$. Write (forgetting for simplicity the indices v in the rest of the proof)

$$\pi_v = \pi = \sigma_1 \boxplus \cdots \boxplus \sigma_m$$

σ_i being the square-integrable representation of $GL(2, \mathbb{R})$ associated to $(z^{a_{2i-1}}\bar{z}^{a_{2i}}, z^{a_{2i}}\bar{z}^{a_{2i-1}})$.

We will use Theorem 3.3 of Borel-Wallach [14, Ch. III]. Let $P = MN$ be the parabolic, of type (2,2,...,2), associated to π. Let A_P be the connected split component (for the real topology) of $M : A_P \cong (\mathbb{R}^\times_+)^m$. We have the Langlands decomposition $M = {}^0MA_P$, with ${}^0M \cong (SL_2^\pm)^m$, where $SL_2^\pm = \{g \in GL(2, \mathbb{R}) : \det g = \pm 1\}$. We identify $\operatorname{Lie}(A_P) \otimes \mathbb{C} = \mathfrak{a}_P \cong \mathbb{C}^m$. To conform to the notations of Borel-Wallach, set:

$\sigma =$ the representation of ${}^0M$ obtained by restriction from $\sigma_1 \otimes \cdots \otimes \sigma_m$.

$\nu =$ the character of A_P obtained by restricting the central character of $\sigma_1 \otimes \cdots \otimes \sigma_m$.

The pair (σ, ν) determines a representation of M, and

$$\pi = \operatorname{Ind}_P^G(\sigma \otimes \nu \otimes 1).$$

Let $\mathfrak{t}$ be a Cartan subalgebra of ${}^0\mathfrak{m} \otimes \mathbb{C}$; $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}_p$ is then a Cartan subalgebra of $\mathfrak{g}$ (we take for $\mathfrak{t}$ the diagonal subalgebra, so that $\mathfrak{h}$ is the diagonal subalgebra of $\mathfrak{g}$). Finally, we will denote by λ the highest weight (for the order of roots on $\mathfrak{b}$ already used) of a rational representation of G. We denote this representation by V^λ.

Let $\lambda_\sigma \in \mathfrak{t}^*$ be the Harish-Chandra parameter of σ: it is defined modulo the complex Weyl group of ${}^0\mathfrak{m}$. We consider λ_σ such that $\langle \lambda_\sigma, \alpha \rangle < 0$ for every positive root α of ${}^0\mathfrak{m}$ with respect to $\mathfrak{t}$, for the order induced by that fixed on the roots of $(\mathfrak{g}, \mathfrak{h})$.

Let $W = W(\mathfrak{g}, \mathfrak{h})$, $W_M = W({}^0\mathfrak{m}, \mathfrak{t}) = W(\mathfrak{m}, \mathfrak{h})$. We have the Kostant decomposition

$$W = W_M W^P$$

where $W^P = \{s \in W : s^{-1}\alpha > 0, \forall \alpha \text{ simple root of } (\mathfrak{m}, \mathfrak{h})\}$.

That said, according to [14, Thm. 3.3, Ch. III],

(a) If $H^\bullet(\mathfrak{g}, K, \pi \otimes V^\lambda) \neq 0$, there exists a unique $s \in W^P$ such that

$$-(\delta + \lambda) = s^{-1}(\lambda_\sigma + \nu)$$

(b) If the element s of W^P satisfies (a),

$$H^{\bullet+\ell(s)}(\mathfrak{g}, K; \pi \otimes V^\lambda) \cong H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}) \otimes \Lambda^\bullet \mathfrak{a}_p^*.$$

On the right-hand side, $s(\lambda + \delta) - \delta$ is the highest weight of a rational representation $V^{s(\lambda+\delta)-\delta}$.

We apply this to the cohomology of $\pi \otimes \tilde{V}^m$; we thus have $\tilde{V}^m = V^\lambda$, with $\lambda = -w_0 m$, w_0 being the element of greatest length of W. We have:

$$\begin{aligned} \lambda_\sigma + \nu &= (a_1, a_2, a_3, a_4, \ldots, a_{2m}); \\ -(\delta + \lambda) &= -(\delta - w_0 m) = w_0(\delta + m) \\ &= w_0(a_2, a_4, \ldots, a_{2m}, a_{2m-1}, \ldots, a_1) \\ &= (a_1, a_3, \ldots, a_{2m-1}, a_{2m}, \ldots, a_2). \end{aligned}$$

We therefore indeed have

$$\lambda_\sigma + \nu = -s(\delta + \lambda)$$

with

$$s^{-1} = \begin{pmatrix} 1 & 3 & 5 & \ldots & 2m-1, & 2m, & \ldots, & 2 \\ 1 & 2 & 3 & \ldots & & & & 2m \end{pmatrix},$$

i.e.

$$s^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & 2m-1, & 2m \\ 1 & 2m & 2 & 2m-2 & 3 & \ldots & m, & m+1 \end{pmatrix}$$

(note that $s \in W^P$). Condition (a) is therefore satisfied, and by (b):

$$H^\bullet(\mathfrak{g}, K; \pi \otimes \tilde{V}^m) = H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}) \otimes \Lambda^\bullet \mathfrak{a}_p^*.$$

We have $s(\lambda + \delta) - \delta = -\lambda_\sigma - \nu - \delta$, i.e.,

$$s(\lambda + \delta) - \delta = (-a_1 - \frac{n-1}{2}, -a_2 - \frac{n-3}{2}, \ldots, -a_n + \frac{n-1}{2}).$$

By the Künneth formula applied to ${}^0M = (SL(2)^{\pm})^m$, one must compute the terms

$$H^\bullet(\mathfrak{sl}(2), 0(2); \sigma_i \otimes V_i)$$

where V_i is the rational representation of $GL(2)$ of highest weight

$$(-a_{2i-1} - \frac{n-1}{2} + 2i - 2, -a_{2i} - \frac{n-1}{2} + 2i - 1).$$

We know, from the calculation for $GL(2)$, that

$$H^\bullet(\mathfrak{sl}(2), 0(2); \sigma_i \otimes (V^{a_{2i}-\frac{1}{2}, a_{2i-1}+\frac{1}{2}})) \cong \Lambda^{\bullet-1}\mathbb{C}^0,$$

and V_i is obtained from the indicated representation by twisting by the character

$$(\det x)^{a_{2i}+a_{2i-1}+\frac{n}{2}-1-2i+2}$$

(note that $\frac{n}{2} = m \in \mathbb{Z}, a_{2i} + a_{2i-1} \in \mathbb{Z}$). Since the representation σ_i, restricted to $SL(2)^{\pm}$, is stable under twisting by $\det x$ (because σ_i belongs to the discrete series), the cohomology does not change. We therefore obtain

$$H^{\bullet+\ell(s)}(\mathfrak{g}, K; \pi \otimes \tilde{V}^m) \cong \otimes^m \Lambda^{\bullet-1}\mathbb{C}^0 \otimes \Lambda^\bullet \mathbb{C}^m \cong \Lambda^{\bullet-m}\mathbb{C}^m.$$

We have $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$; the central character of $\pi \otimes \tilde{V}^m$, restricted to $A = A_G$, is trivial; the Künneth formula for $\tilde{\mathfrak{g}} \times \mathfrak{a}$ then gives the result in the even case, up to a shift; by Borel-Wallach [14, Ch. IV, Thm. 5.1], we have $\ell(s) = m(m-1)$, whence Lemma 3.14 in this case.

Consider now the case where $n = 2m+1$ is **odd**. The representation π is therefore associated to the characters

$$(z^{a_1}\bar{z}^{a_2}, z^{a_2}\bar{z}^{a_1}, \ldots, z^{a_{2m}}\bar{z}^{a_{2m-1}}, (z\bar{z})^{a_{2m+1}})$$

with

$$a_1 < a_2, \quad a_3 < a_4, \quad \ldots,$$
$$a_1 + a_2 = \cdots = 2a_{2m+1} = w + 1 - n.$$

We again assume $a_1 < a_3 < \ldots$, whence finally:

$$a_2 > a_4 > \cdots > a_{2m} > a_{2m+1} > a_{2m-1} > \cdots > a_1.$$

The representation π is induced, from the parabolic $P = MN$ of type (2,2,...,2,1), from the representation

$$\sigma = \sigma_1 \otimes \cdots \otimes \sigma_m \otimes \chi$$

where σ_i is the obvious discrete series, and χ a character of $\mathbb{R}$ of the form $x \mapsto |x|^{a_{2m+1}}(\operatorname{sgn} x)^\epsilon$. We use notations analogous to the even case: thus $M = {}^0MA_M, \mathfrak{a}_M \cong \mathbb{C}^{m+1}$, ${}^0M \cong (SL_2^{\pm})^m \times \{\pm 1\}$.

We have, still with the notations introduced in the even case:

$$p = (a_2 - \frac{n-1}{2}, a_4 - \frac{n-1}{2}, \dots, a_1 - \frac{n-1}{2})$$

$$m = (a_2 - \frac{n-1}{2}, a_4 - \frac{n-3}{2}, \dots, a_1 + \frac{n-1}{2})$$

$$\lambda = -w_0 m = (-a_1 - \frac{n-1}{2}, -a_3 - \frac{n-3}{2}, \dots, -a_2 + \frac{n-1}{2})$$

$$\lambda_\sigma + \nu = (a_1, a_2, \dots, a_{2m+1})$$

$$\delta + \lambda = (-a_1, -a_3, \dots, -a_2).$$

We must write $-(\delta + \lambda) = s^{-1}(\lambda_\sigma + \nu)$, whence

$$s^{-1} = \begin{pmatrix} 1 & 3 & 5 & 7 \dots & 2m-1, & 2m+1, & 2m, & \dots, & 4 & 2 \\ 1 & 2 & 3 & \dots & m, & m+1, & m+2, & \dots, & 2m & 2m+1 \end{pmatrix}$$

i.e.,

$$s^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2m-1 & 2m & 2m+1 \\ 1 & 2m+1 & 2 & 2m & \dots & m & m+2 & m+1 \end{pmatrix}$$

(note that $s \in W^P$). Using again part (b) of the Borel-Wallach theorem, we are led to compute

$$H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}).$$

We have

$$\begin{aligned} s(\lambda+\delta) - \delta &= -\lambda_\sigma - \nu - \delta \\ &= (-a_1 - \frac{n-1}{2}, -a_2 - \frac{n-3}{2}, \dots, -a_{2m+1} + \frac{n-1}{2}). \end{aligned}$$

Using the Künneth formula for ${}^0M$, we are led to first compute the terms in

$$H^\bullet(\mathfrak{sl}(2), 0(2); \sigma_i \otimes V_i)$$

where V_i is described in the proof of the even case; we know that each of these factors is equal to $\Lambda^{\bullet-1}\mathbb{C}^0$.

It remains to compute

$$H^\bullet(0, \{\pm 1\}; \chi \otimes x^{-a_{2m+1} + \frac{n-1}{2}})$$

(sic), where, recall, $\chi(x) = |x|^{a_{m+1}}(\text{sgn } x)^\epsilon$; we twist by the rational character of $GL(1)$:

$$x \mapsto x^{-a_{2m+1} + \frac{n-1}{2}}$$

(note that $a_{2m+1} + \frac{n-1}{2} \in \mathbb{Z}$); we consider here the $(\mathfrak{g}, K)$-cohomology for $\mathfrak{g} = \{0\}, K = \{\pm 1\} \subset \mathbb{R}^\times$. This space is equal to $\mathbb{C}$ in degree zero if $\chi(x)x^{-a_{2m+1}+\frac{n-1}{2}}$ is an **even** character of $^\times$, zero otherwise.

In the first case, we therefore obtain a term in $\Lambda^\bullet\mathbb{C}^0$; multiplying by the terms already obtained, we have

$$H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes V^{s(\lambda+\delta)-\delta}) \cong \otimes^m \Lambda^{\bullet-1}\mathbb{C}^0 \otimes \Lambda^\bullet\mathbb{C}^0 \otimes \Lambda^\bullet\mathbb{C}^{m+1} \cong \Lambda^{\bullet-m}\mathbb{C}^{m+1}.$$

By Borel-Wallach [14, Thm. 5.1 p. 101], we have $\ell(s) = m^2$, whence

$$H^\bullet(\mathfrak{g}, K; \pi \otimes \tilde{V}^m) \cong \Lambda^{\bullet-m(m+1)}\mathbb{C}^{m+1}.$$

Dividing, as in the even case, by the term $\Lambda^\bullet\mathfrak{a} = H^\bullet(\mathfrak{a}, \mathbb{C})$ associated to the decomposition $\mathfrak{g} = \tilde{\mathfrak{g}} \times \mathfrak{a}$, we deduce the desired result.

Suppose now that $\chi(x)x^{-a_{2m+1}+\frac{n-1}{2}}$ is an odd character. Let ϵ be the non-trivial character of order 2 of $\mathbb{R}^\times$. We then have

$$H^\bullet({}^0\mathfrak{m}, K_M; \sigma \otimes \epsilon \otimes V^{s(\lambda+\delta)-\delta}) \cong \Lambda^{\bullet-m}\mathbb{C}^{m+1},$$

$\sigma \otimes \epsilon$ denoting the representation $(\sigma_1 \otimes \epsilon, \ldots, \sigma_m \otimes \epsilon, \chi_\epsilon)$ of M (or ${}^0M$), where we twist all factors by the character ϵ of the determinant (note that $\sigma_i \cong \sigma_1 \otimes \epsilon$). This representation induces to the representation $\pi \otimes \epsilon$ of $GL(n, \mathbb{R})$. We therefore finally have $H^\bullet(\mathfrak{g}, K, \pi \otimes \epsilon \otimes \tilde{V}^m) \cong \Lambda^{\bullet-m(m+1)}\mathbb{C}^{m+1}$, which completes the proof.

Remark. If n is **even**, we see that there is no need to twist by a character.

To prove Theorem 3.13, we now consider, following Harder [30 a, c], the arithmetic quotients of $X = G(F_\infty)/K_\infty$, where $K_\infty = \prod_{v \text{ infinite}} K_v$, by congruence subgroups Γ. For every open compact subgroup $K \subset G(\mathbb{A}_f)$, we have

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K \cong \coprod_i \Gamma_i \backslash X$$

(finite union, the Γ_i being congruence subgroups). We set $S_K = \coprod_i \Gamma_i\backslash X$; we have

$$\tilde{S} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty = \varprojlim_K S_K$$

with the obvious morphisms; we **define**

$$H^\bullet(\tilde{S}, \mathbb{C}) = \varinjlim H^\bullet(S_K, \mathbb{C}).$$

More generally, a rational representation (τ, V) of $L = \mathrm{Res}_{F/\mathbb{Q}} GL(n)$ defines a system of coefficients $\mathcal{V}$ on S, and systems of coefficients, also denoted $\mathcal{V}$, on the S_K. We **define**

$$H^\bullet(\tilde{S}, \mathcal{V}) = \varinjlim H^\bullet(S_K, \mathcal{V}).$$

The group $G(\mathbb{A}_f)$ acts naturally on $H^\bullet(\tilde{S},\mathcal{V})$. For every finite level K, let

$$H^\bullet_{\rm cusp}(S_K,\mathcal{V}) = \bigoplus H^\bullet_{\rm cusp}(\Gamma_i\backslash X,\mathcal{V}).$$

the parabolic cohomology [10 c]. We set, **by definition**:

$$H^\bullet_{\rm cusp}(\tilde{S},\mathcal{V}) = \varinjlim H^\bullet_{\rm cusp}(S_K,\mathcal{V}).$$

By a fundamental result of Borel [10 c], the parabolic cohomology, at finite levels, injects into $H^\bullet(S_K,\mathcal{V})$. Passing to the inductive limit, we deduce:

Lemma 3.15.

(i) We have $H^\bullet_{\rm cusp}(\tilde{S},\mathcal{V}) \hookrightarrow H^\bullet(\tilde{S},\mathcal{V})$.

(ii) $H^\bullet_{cusp}(\tilde{S},\mathcal{V}) = \bigoplus_\pi H^\bullet(\mathfrak{g}_\infty,K_\infty;\pi_\infty\otimes\mathcal{V})\otimes\pi_f$; *the sum is over the submodules* π *of* $\mathcal{A}_{\rm cusp}(G(F)\backslash G(\mathbb{A}_F))$ *whose central character coincides, on the identity component* $Z(F_\infty)^+$, *with that of* $\tilde{\mathcal{V}}$.

(we have set $\mathfrak{g}_\infty = \bigotimes_{v\,\text{infinite}}\mathfrak{g}_v$*)*

Let E be the field of rationality of the representation (τ,V). Note that (τ,V) is in fact defined over E. Indeed, τ is defined by its dominant weights $(m_\iota)_{\iota\in I}$; we thus have $m_\iota = (m_\iota^1,\dots,m_\iota^n)$, $m^1\geq\ldots\geq m^n$. The field E is the extension of $\mathbb{Q}$ fixed by the subgroup Γ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ defined by the equations $m_{\sigma\iota} = m_\iota (\sigma\in\Gamma,\iota\in I)$. In particular, $m^i_{\sigma\iota} = m^i_\iota$ and the character of $GL(1,F)$ defined by $x\mapsto\Pi(x_\iota)^{m^i_\iota}$ is thus defined over E, and therefore takes values in E. The same is true for the character of the maximal torus $GL(1,F)^n$ of $GL(n,F)$ defined by (m_ι). Since this character appears with multiplicity 1 in V, τ is defined over E by Waldspurger's Lemma 1.1 [58].

Consequently, the transition morphisms of the system of coefficients $\mathcal{V}$ are given by matrices with coefficients in E. We deduce that the Betti cohomology $H^\bullet_B(S_K,\mathcal{V})$, and its projective limit $H^\bullet_B(\tilde{S},\mathcal{V})$ are naturally defined over E. We denote by $H^\bullet_B(\tilde{S},\mathcal{V}_E)$ this cohomology space; we have

$$H^\bullet(\tilde{S},\mathcal{V}) = H^\bullet_B(\tilde{S},\mathcal{V}_E)\otimes_E\mathbb{C}.$$

The action of $G(\mathbb{A}_f)$ on $H^\bullet(\tilde{S},\mathcal{V})$ by Hecke correspondences preserves this E-structure. For the moment, we do not know if the subspace $H^\bullet_{\rm cusp}\subset H^\bullet$ is defined over E. Nevertheless:

Proposition 3.16. *(Drinfeld-Manin) Let W be an irreducible subquotient of $H^\bullet(\tilde{S},\mathcal{V})$ for the action of $G(\mathbb{A}_f)$. Then there exists a finite extension E' of E such that W is rational over E'.*

Note immediately that the Proposition implies the first part of the theorem: if π is regular, we have constructed a system of coefficients τ such that $H^\bullet(\mathfrak{g}_\infty, K_\infty; \pi_\infty \otimes \epsilon_\infty \otimes \tau_\infty) \neq 0$ (Lemma 3.14), ϵ_∞ being a character of order 2 of $F_\infty^\times$, which we extend to an Artin character of order 2, ϵ, of $\mathbb{A}_F^\times$[3]. Lemma 3.15 produces an irreducible subspace of $H^\bullet(\tilde{S}, \mathcal{V})$ isomorphic to $\pi_f \otimes \epsilon$ as a representation of $G(\mathbb{A}_f)$, whence the result.

Proof. [**of Proposition 3.16**] Note that $H^\bullet(\tilde{S}, \mathcal{V})$ is a smooth and admissible $G(\mathbb{A}_f)$-module. In particular, W is an irreducible admissible representation of $G(\mathbb{A}_f)$. Let $K \subset G(\mathbb{A}_f)$ such that $W^K \neq \{0\}$. We can further assume that $K = \prod_{v \text{ finite}} K_v$. Let $\mathcal{H} = C_c^\infty(G(\mathbb{A}_f)//K)$. We know (cf. [5]) that $\mathcal{H}$ acts irreducibly on W^K, and that this action determines the isomorphism class of W. The algebra $\mathcal{H}$ is defined over $\mathbb{Q}$; by paragraph 3.1, the field of rationality of W is then that of this representation of $\mathcal{H}$. Consider the space $T_E = H_B^\bullet(\tilde{S}, \mathcal{V}_E)^K$; this is the space of a representation of $\mathcal{H}$, defined over E. Let $T_\mathbb{C} = T_E \otimes_E \mathbb{C}$. By Burnside's theorem, the irreducible subquotients of $T_\mathbb{C}$ (as an $\mathcal{H}(\mathbb{C})$-module) or of $T_{\bar{\mathbb{Q}}} = T_E \otimes_E \overline{\mathbb{Q}}$ (as an $\mathcal{H}(\overline{\mathbb{Q}})$-module) coincide. Consequently, W^K is obtained by extension of scalars from one of the irreducible subquotients of $T_\mathbb{Q}$, and this one can be defined over a finite extension E'. □

Remark. Even if W is a submodule, this does not imply that it is of the form $W_{E'} \otimes_{E'} \mathbb{C}$ with $W_{E'} \in H_B^\bullet(\tilde{S}, \mathcal{V}_E) \otimes E'$; this is false in general, cf. [30 b].

The rest of Theorem 3.13 is more difficult to prove. To simplify, suppose first $F = \mathbb{Q}$. The representation (τ, V) can then be chosen defined over $\mathbb{Q}$.

Let $H_c^\bullet(S_K, \mathcal{V})$ be the compactly supported cohomology space of S_K in $\mathcal{V}$. If $\overline{S_K}$ is the Borel-Serre compactification [13, 47] of S_K, ∂S_K its boundary, we have an exact triangle:

$$\begin{array}{ccc} H_c^i(S_K, \mathcal{V}) & \longrightarrow & H^i(\overline{S_K}, \mathcal{V}) = H^i(S_K, \mathcal{V}) \\ & \overset{\delta}{\nwarrow} \quad \swarrow & \\ & H^i(\partial\overline{S_K}, \mathcal{V}) & \end{array}$$

the homomorphism δ being of degree $+1$. In particular, we see that $H_c^\bullet(S_K, \mathcal{V})$ is defined over $\mathbb{Q}$ in the Betti cohomology.

We know that cuspidal cohomology classes have zero restriction to $\partial\overline{S}_K$ [47, Satz 1.10], whence natural injections

$$H_{\text{cusp}}^i(S_K, \mathcal{V}) \to H_c^i(S_K, \mathcal{V}).$$

Let $d = \dim(A\backslash X)$, where A is the topological identity component of $Z(\mathbb{R})$: $A \cong \mathbb{R}_+^\times$. The natural map $\Gamma\backslash X \to A\Gamma\backslash X$ is a homotopy equivalence; for Γ

arithmetic, $A\Gamma\backslash X$ has finite volume. Set $X = A\backslash X$[7], $S_K^1 = AG(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K$. In the preceding descriptions, we can everywhere replace S_K by S_K^1. Note that $G(\mathbb{A}) = AG(\mathbb{A})^1$, where $G(\mathbb{A})^1$ is the subgroup of $G(\mathbb{A})$ annihilated by characters of the form $|\chi(g)|$ where $\chi : G \to \mathbb{G}_m$ is a rational character. We therefore have direct decompositions $S_K = AS_K^1$, with

$$S_K^1 = G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_\infty K.$$

If (τ^*, V^*) is the dual representation of (τ, V), $\mathcal{V}^*$ the associated local system, we have a natural, non-degenerate duality

$$H_c^i(S_K^1, V) \times H^{d-i}(S_K^1, \mathcal{V}^*) \to \mathbb{C}.$$

It is defined over $\mathbb{Q}$ (for the Betti structures). If $\tilde{\omega}, \tilde{\eta}$ are two differential forms representing, in De Rham cohomology, classes $\omega \in H_c^i, \eta \in H^{d-i}$, the duality is given by

$$\langle \omega, \eta \rangle = \int_{S_K^1} \langle \tilde{\omega}(x), \tilde{\eta}(x) \rangle.$$

Restricted to $H^i_{\text{cusp}} \times H^{d-i}_{\text{cusp}} \subset H_c^i \times H^i$, the duality is still non-degenerate. From the automorphic point of view, it is expressed as follows: let $\omega \in H^i_{\text{cusp}}(S_K, \mathcal{V})$, represented by a differential form

$$\tilde{\omega} \in \text{Hom}_{K_\infty}(\Lambda^i \tilde{\mathfrak{p}}, \mathcal{A}_{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K) \otimes V);$$

we have set $\mathfrak{p} = \mathfrak{g}_\infty/\mathfrak{k}_\infty, \tilde{\mathfrak{p}} = \mathfrak{p}/\mathfrak{a}$ where $\mathfrak{a} = \text{Lie}(A)$. Similarly let $\tilde{\eta}$ represent $\eta \in H^{d-i}_{\text{cusp}}(S_K, \mathcal{V}^*)$. Then

$$\tilde{\omega} \wedge \tilde{\eta} \in \text{Hom}_{K_\infty}(\Lambda^d \tilde{\mathfrak{p}}, C^\infty(S_K))$$

defines a differential form with rapid decay on S_K. Its integral over S_K^1 is equal to $\langle \omega, \eta \rangle$.

We now consider the spaces of reduced L^2-cohomology [61] of the varieties S_K^1. They are defined as follows: let $\Omega^i_{(2)}(S_K^1, \mathcal{V})$ be the space of i-forms φ with values in V which are square-integrable, as well as their derivative $d\varphi$ (defined as a distribution), for the invariant measure on S_K^1 (we have endowed V and $\Lambda^i \tilde{\mathfrak{p}}$ with admissible scalar products, cf. [11, 61]); $H^\bullet_{(2)}(S_K^1, V)$ is the cohomology of this complex, and

$\tilde{H}^i_{(2)}(S_K^1, V)$, the reduced cohomology, is defined by

$$\tilde{H}^i_{(2)}(S_K, V) = \{\varphi \in \Omega^i_{(2)} : d\varphi = 0\}/\overline{d\Omega^i_{(2)}}$$

[7] [TRANS] LHS is a typo.

where the closure is for the L^2 topology on the i-forms. We have a natural map, realizing the cohomology with supports in De Rham cohomology:

$$j : H^i_c(S^1_K, \mathcal{V}) \to H^i_{(2)}(S^1_K, \mathcal{V}).$$

The following lemma, though simple, is fundamental:

Lemma 3.17. *Let $\alpha \in H^i_c(S^1_K, \mathcal{V})$ and $\beta \in H^{d-i}_c(S^1_K, \mathcal{V}^*)$ such that $\langle \alpha, \beta \rangle \neq 0$. Then $j\alpha \neq 0$, $j\beta \neq 0$.*

Proof. We define a linear form on $\tilde{H}^i_{(2)}(S_K, \mathcal{V})$ by

$$\lambda_\beta : \alpha \mapsto \langle \alpha, \beta \rangle = \int_{S^1_K} (\alpha, \beta)$$

where (α, β) is the differential form (with trivial coefficients) obtained using the duality between V and V^*. Let us show that this is well-defined. First, if $\alpha \in \Omega^i_{(2)}$, (α, β) is compactly supported and square-integrable, so the integral is well-defined. Suppose that $\alpha = d\eta$, with $\eta \in \Omega^{i-1}_{(2)}$. We have the formula

$$d((\alpha, \beta)) = (d\alpha, \beta) + (-1)^{\deg \alpha}(\alpha, d\beta),$$

whence

$$\langle d\eta, \beta \rangle = \int_{S^1_K} [d((\eta, \beta)) + (-1)^{\deg \eta + 1}(\alpha, d\beta)].$$

Since $d\beta = 0$, it suffices to show that the first integral vanishes. Let $\omega = (\eta, \beta)$. This is a compactly supported differential form, square-integrable, and such that $d\omega = (d\eta, \beta)$ is square-integrable since $\eta \in \Omega^{i-1}_{(2)}$. We must show that $\int_{S^1_K} d\omega = 0$. Using a partition of unity, we can assume ω is supported in a system of local coordinates $(x_1, \ldots, x_d)$; we thus have

$$\omega = \sum \omega_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_d, \quad d\omega = \sum (-1)^i \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_d,$$

and $\int d\omega = \sum (-1)^i \int \frac{\partial \omega_i}{\partial x_i} dx_1 \ldots dx_n$. If F is a C^∞ function with compact support, equal to 1 on the support of ω, each term is then equal to $(-1)^i(-1) \int \left(\frac{\partial F}{\partial x_i} \omega_i \right)$ by definition of the distributional derivative $\frac{\partial \omega_i}{\partial x_i}$. This integral is zero, whence the result.

We have therefore shown that λ_β defines a linear form on the non-reduced cohomology space $H^i_{(2)}(S^1_K, V)$. It remains to see that λ_β passes to the quotient to define a linear form on $\tilde{H}^i_{(2)}$. By definition of this space, it suffices to verify that

λ_β is continuous for the topology on $\Omega^i_{(2)}$ given by the scalar product on i-forms. This is clear: suppose $(\Lambda^i\tilde{\mathfrak{p}})^*, V$ equipped with their admissible scalar products; let (λ_a, e_b) be orthonormal bases, and (λ'_a, e'_b) dual bases of $(\Lambda^{d-i}\mathfrak{p})^*, V$.

If $\alpha \in \Omega^i_{(2)}(S^1_K, V)$, we can write

$$\alpha = \sum_{a,b} f_{ab}\lambda_a \otimes e_b \in \mathrm{Hom}_K(\Lambda^i\tilde{\mathfrak{p}}^*, L^2(S^1_K)\otimes V)$$

with $f_{a,b} \in L^2(S^1_K)$. Similarly, $\beta = \sum_{a,b} g_{a,b}\lambda'_a \otimes e'_b$ with $g_{ab} \in C^\infty_c(S^1_K)$.

We have

$$\langle \alpha, \beta\rangle = \sum_{a,b}\int_{S^1_K} f_{ab}g_{ab}dx$$

which is of course continuous for the L^2 structure. The form λ_β is therefore well-defined on $\tilde{H}^i_{(2)}$. If it does not vanish on α, we therefore have $j\alpha \neq 0$.

□

Remark. Let $H^1_!$ be the image of H^i_c in H^i. Since H^i and H^{d-i}_c are in non-degenerate duality, $H^i_!$ is the quotient of H^i_c by the kernel E^i_c of the pairing form on $H^i_c \times H^{d-i}_c$. Lemma 3.17 therefore has the following consequence; the proof applies to any reductive group G defined over a number field F and to any system of coefficients $\mathcal{V}$ coming from an algebraic representation of G with complex coefficients; we again set $S^1_K = G(\mathbb{Q})\backslash G(\mathbb{A})^1/K_\infty K_f$:

Proposition 3.18. *We have a canonical injection*

$$H^i_!(S^1_K, \mathcal{V}) \hookrightarrow \tilde{H}^i_{(2)}(S^1_K, \mathcal{V}).$$

This theorem was proved, in a slightly different way, by Harder [30 a]. Moreover, since the duality restricted to H^i_{cusp} is non-degenerate, we know that the natural map

$$H^i_{\mathrm{cusp}}(S^1_K, \mathcal{V}) \hookrightarrow H^i_!(S^1_K, \mathcal{V})$$

is also an injection.

Let then π be the representation considered in Theorem 3.13. Suppose first that Lemma 3.14 is verified with $\epsilon_\infty = 1$. We have seen that π_f identifies with a submodule of $H^i_{\mathrm{cusp}}(\tilde{S}^1, \mathcal{V})$. For K deep enough, the representation of the Hecke algebra $\mathcal{H} = C^\infty_c(G(\mathbb{A}_f)/K)$ associated to π_f (cf. Prop. 3.16) therefore appears in $H^i_{\mathrm{cusp}}(S^1_K, \mathcal{V})$ and thus in $H^i_!(S^1_K, \mathcal{V})$. Since this space is defined over $\mathbb{Q}$, the representation $\sigma_{\pi f}$, for $\sigma \in \mathrm{Aut}(\mathbb{C})$ therefore also appears as a submodule of $H^i_!(S^1_K, \mathcal{V})$ and thus of $H^i_{(2)}(S^1_K, \mathcal{V})$ by Proposition 3.18.

By Borel and Casselman [11], we have

$$H^i_{(2)}(S^1_K, \mathcal{V}) = \oplus_\xi H^i(\mathfrak{g}, K_\infty; \xi_\infty \otimes V)\otimes \xi_f^K. \tag{3.9}$$

[8] where ξ runs through a complete system of submodules of the discrete spectrum $L^2_{dis}(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$. Note that an irreducible representation of $G(\mathbb{A})$ identifies with a representation of $G(\mathbb{A})^1$, once its central character on A is known; in particular, if we fix it equal to the central character $\omega_\pi|_A$, a representation ξ of $G(\mathbb{A})^1$ determines a representation ξ_f of $G(\mathbb{A}_f)$. This is the action of $G(\mathbb{A}_f)$ on $\overline{H}^i_{(2)}(S^1_K, \mathcal{V})$ obtained using this description.

Let then ξ be a representation appearing in (3.6), whose central character ω_ξ satisfies $\omega_\xi|_A = \omega_\pi|_A$, and such that $\xi_f \cong \sigma_{\pi f}$. Let ξ_∞ be its component at infinity, and let

$$(z^{r_1}(\bar{z})^{r_2}, z^{r_2}\bar{z}^{r_1}, \ldots, z^{r_{2k}}\bar{z}^{r_{2k-1}}, (z\bar{z})^{r_{2k+1}}, \ldots, (z\bar{z})^{r_n})$$

be the characters of $\mathbb{C}^\times$ associated to ξ_∞ by the Langlands classification, with $r_i \neq r_j$ $(i \leq 2k)$.

The condition $H^i(\tilde{\mathfrak{g}}, K_\infty; \xi_\infty \otimes V) \neq 0$ implies that $\xi_\infty|_{\tilde{\mathfrak{g}}}$ has the same infinitesimal character as $\pi_\infty|_{\tilde{\mathfrak{g}}}$, whence

$$(r_1, \ldots, r_n) = w(a_1 + s, \ldots, a_n + s) \quad (w \in \mathfrak{S}_n)$$

for an $s \in \mathbb{C}$. By the condition on the central character, we moreover have $\sum a_i = \sum r_i$, so $s = 0$. If r is the representation of $W_\mathbb{C} = \mathbb{C}^\times$ associated to ξ_∞, we see therefore that $r|\ |_\mathbb{C}^{\frac{1-n}{2}}$ is a sum of algebraic characters (**Attention:** this does not imply that ξ_∞ and π_∞ are isomorphic: think of the representations of $GL(2,\mathbb{R})$ associated to the characters $(z\bar{z}|\ |^{\frac{1}{2}}, |\ |^{\frac{1}{2}})$ and $(z|\ |^{\frac{1}{2}}, \bar{z}|\ |^{\frac{1}{2}})$ of $\mathbb{C}^\times$).

The theory of Eisenstein series [38 a, f] implies that ξ is obtained using Eisenstein series formed from a cuspidal representation of a parabolic P of type $(n_1, \ldots, n_r)$; ξ is then isomorphic to a subquotient of an induced representation ρ which can be normalized à la[9] Tate:

$$\rho = \left[\mathcal{I}\mathrm{nd}_{P(\mathbb{A})}^{G(\mathbb{A})}(\xi_1|\ |^{\frac{1-n_1}{2}} \otimes \cdots \otimes \xi_r|\ |^{\frac{1-n_r}{2}})\right] \cdot |\ |^{\frac{n-1}{2}}, \tag{3.10}$$

[10] the ξ_i being cuspidal representations of $GL(n_i, \mathbb{A})$. If r_i is the representation of $W_\mathbb{C}$ associated by Langlands to $(\xi_i)_\infty$, r' that associated to the Langlands subquotient of ρ_∞, we then have

$$r' \otimes |\ |_\mathbb{C}^{\frac{1-n}{2}} \cong \oplus_i r_i \otimes |\ |_\mathbb{C}^{\frac{1-n_i}{2}}.$$

The previous considerations on the central character and the infinitesimal character show that $r' \otimes |\ |_\mathbb{C}^{\frac{1-n}{2}}$ and thus the $r_i \otimes |\ |_\mathbb{C}^{\frac{1-n_i}{2}}$, are algebraic. Consequently, $\xi_i \in Alg(n_i)^0$. Moreover, ξ_i is regular (infinitesimal character !).

[8] [TRANS] In the original paper, the label of this equation is also (3.6).
[9] [TRANS] In the style or manner of sth.
[10] [TRANS] In the original paper, the label of this equation is also (3.7)

Suppose then $P \neq G$. By induction, we can apply Theorem 3.13 to the ξ_i. Let $\alpha = \sigma^{-1}$. There therefore exist cuspidal representations ${}^{\alpha}\xi_i$, whose finite part is ${}^{\alpha}\xi_{i,f}$. Since the induction (3.7) is, for the finite part of the representations, a rational operation (Lemma 3.9), ${}^{\alpha}\xi_f = \pi_f$ would then be a subquotient of ${}^{\alpha}\rho_f$. This contradicts Theorem 1.1.

We see therefore that ξ is cuspidal. The considerations on the infinitesimal character and the central character imply that the type at infinity of ξ is equal to that of π, whence Theorem 3.13 for π. Finally, in general, if π satisfies Lemma 3.14 with a non-trivial character ϵ_∞, we extend it to an Artin character of order 2, ϵ, of $\mathbb{A}^\times$: Theorem 3.13 for π is deduced from the assertion for $\pi \otimes \epsilon$.

Let us briefly indicate how to extend the argument to an arbitrary field F. Let $\pi \in Alg^0(n, F)$ be a regular representation, (τ, V) the representation of $L = \mathrm{Res}_{F/\mathbb{Q}} GL(n)$ associated to π. We can assume it defined over $\overline{\mathbb{Q}}$; we have $\tau = \otimes_{\iota \in I} \tau_\iota$. Let $\sigma \in \mathrm{Aut}\,\mathbb{C}$; we also denote by σ its image in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ where we consider $\overline{\mathbb{Q}}$ as embedded in $\mathbb{C}$. We can consider the abstract representation $({}^{\sigma}\tau, {}^{\sigma}V)$ of $L(\mathbb{Q})$ on $V \otimes_{\mathbb{C}, \sigma^{-1}} \mathbb{C}$ (cf. 3.1). It is clear that it is obtained by extension of scalars to $\mathbb{C}$ of the rational representation, still denoted ${}^{\sigma}\tau$, of $H(\mathbb{Q})$: ${}^{\sigma}\tau = \otimes_\iota \tau_{\sigma^{-1}\iota}$.

The simplicial realization of Betti cohomology shows that there is a σ-linear isomorphism:

$$H^\bullet(S_K, \mathcal{V}) \to H^\bullet(S_K, {}^{\sigma}\mathcal{V})$$

where ${}^{\sigma}\mathcal{V}$ is the system of coefficients deduced from ${}^{\sigma}V$. We have analogous isomorphisms for the spaces $H^\bullet_c(S_K, \mathcal{V})$, $H^\bullet_!(S_K, \mathcal{V})$. Since π_f is a submodule of $H^\bullet_!(S_K, \mathcal{V})$, ${}^{\sigma}\pi_f$ is therefore a submodule of $H^\bullet_!(S_K, \mathcal{V})$. We then proceed as in the case of $\mathbb{Q}$, replacing S_K by $S^1_K = AG(F)\backslash G(\mathbb{A})/K_\infty K \cong G(F)\backslash G(\mathbb{A})^1/K_\infty K$, where $A = \mathbb{R}^\times_+$ is embedded diagonally in $Z(F_\infty) = \prod_{v|\infty} Z(F_v)$, the center of $G(F_\infty)$. We similarly obtain a representation ξ of $G(\mathbb{A})$, appearing in the discrete spectrum, with the same central character under A as π, and such that $\xi_f = {}^{\sigma}\pi_f$. We prove similarly, by considering separately all the infinite places, that ξ is algebraic, then cuspidal. It remains to verify that the type at infinity $p(\xi) = (p_\iota(\xi))_\iota$ is equal to $\sigma(p(\pi)) = (p_{\sigma^{-1}\iota}(\pi))_\iota$. This is clear - consider the infinitesimal character of ξ_∞ - since ξ_∞ has cohomology with coefficients in ${}^{\sigma}\tau = \otimes_\iota \tau_{\sigma^{-1}\iota}$.

Note the following consequence of the proof. If $\tau = \otimes_\iota \tau_\iota$ is an irreducible, rational, representation of L, let E be its field of rationality (or of definition). The system of coefficients $\mathcal{V}$ is therefore defined over E, and we have:

Theorem 3.19. *The subspace $H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) \subset H^i_c(\tilde{S}, \mathcal{V})$ is defined over E. More precisely, we have*

$$H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}) = H^i_{\mathrm{cusp}}(\tilde{S}, \mathcal{V}_E) \otimes_E \mathbb{C}$$

with

$$H^i_{\text{cusp}}(\tilde{S}, \mathcal{V}_E) = H^i_c(\tilde{S}, \mathcal{V}) \cap H^i_{\text{cusp}}(\tilde{S}, \mathcal{V}_E).$$

This is clear: we have $H^i_{\text{cusp}}(\tilde{S}, \mathcal{V}) = \bigoplus_\pi H^i(\mathfrak{g}, K; \pi_\infty) \otimes \pi_f$, where π_f runs through the set of cuspidal representations of type at infinity prescribed by V. Recall the injections

$$H^i_{\text{cusp}} \hookrightarrow H^i_! \hookrightarrow \bar{H}^i_{(2)}.$$

Let $W = H^i(\mathfrak{g}, K, \pi_\infty) \otimes \pi_f$ be the submodule of type π_f of $H^i_{\text{cusp}}(\tilde{S}, \mathcal{V})$ where π_f (and thus π) is fixed. We have $W \subset H^i_!$; for $\sigma \in \text{Aut}(\mathbb{C}/E)$ let ${}^\sigma W$ be its transform. It therefore injects into $\tilde{H}^i_{(2)}(\tilde{S}, \mathcal{V})$ by Lemma 3.17; the previous proof shows that it is in fact contained in the image of H^i_{cusp} in $\bar{H}^i_{(2)}$; we deduce that the image of H^i_{cusp} in $H^i_!$ is invariant by $\text{Aut}(\mathbb{C}/E)$; the theorem then follows from Lemma 3.2.1.

Remark. In the case of $GL(2)$, this theorem is due to Eichler, Shimura, Waldspurger and Harder.

4 Motives and Algebraic Representations

4.1 Motives

4.2 Rational Algebraic Representations

4.3 Motives and Algebraic Representations

4.4 Consequences of Motivic Formalism

4.5 Hasse–Weil Conjecture and Reciprocity Theory for Algebraic Representations

5 The Case of Self-Dual Representations

5.1 Statement of the Theorem

5.2 Remarks

5.3 Sketch of Proof